Centrality-Based Ranking of Paired Comparison

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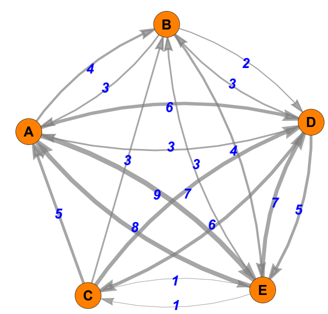
Ranking players or teams is always an interesting topic in sports. For instance, the ATP (Association of Tennis Professionals) tour has more than one thousand active professional players participating in sponsored tournaments around the world. In one season, each player can play against at most dozens of other players, only a small subset of the whole cohort, thus creating a super-sparse ranking network system. It is crucial to extract the intertwined relationships and rank all players based on the limited information available.

Suppose that there are n players or teams who compete in multi-player games or sports. The number of games played between player i and player j is $n_{ij} \geq 0$. Out of these n_{ij} games, player i wins a_{ij} times and player j wins $a_{ji} = n_{ij} - a_{ij}$ times. The win-loss record can be represented by a directed weighted network G whose adjacency matrix $\mathbf{A} = \{a_{ij}\}$. In general $\mathbf{A}$ is asymmetric with zeroes on the diagonal. Figure 1 shows an example of G and $\mathbf{A}$ for a small data set with $n = 5$. The direction of the arrows runs from the winner to the loser with a weight being the frequency of wins. For example, player A defeats player B four times and loses three times.



(a)

$$\mathbf{A} = \begin{pmatrix} 0 & 4 & 0 & 6 & 9 \\ 3 & 0 & 0 & 2 & 4 \\ 5 & 3 & 0 & 7 & 1 \\ 3 & 3 & 6 & 0 & 5 \\ 8 & 3 & 1 & 7 & 0 \end{pmatrix}$$

(b)

Figure 1: Game record (a) and the corresponding adjacency matrix (b). The widths of the edges are proportional to the weights.

For an undirected unweighted network, the eigenvector centrality of a given node is proportional to the sum of the centralities of its neighbors such that $x_i = \lambda \sum_{j=1}^{n} C_{ij} x_j$ where $C_{ij} = 1$ if nodes i and j are connected, and zero otherwise [Bonacich, 1987]. The solution is a list of the centrality measures which can be used to describe the significance of the nodes in the network.

For the directed weighted network in the current situation, a generalization of the eigenvector centrality is defined

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as

$$x_i = \lambda \sum_{j \in \mathrm{Ne}(i)} \frac{a_{ij}}{a_{ij} + a_{ji}} x_j, \tag{1}$$

where x_i is the *score* of player i and $\mathrm{Ne}(i)$ is the set of its neighbors. The basic idea in (1) is score swapping. Two players i and j will exchange some of their scores based on their win-loss record. Player i gains a proportion $\frac{a_{ij}}{a_{ij}+a_{ji}}$ of player j's score, and meanwhile transfers $(1 - \frac{a_{ij}}{a_{ij}+a_{ji}})$ of its own score in return. The factor $\frac{a_{ij}}{a_{ij}+a_{ji}}$ is the estimated probability that player i is preferred to player j in a single game [Bradley and Terry, 1952]. Note that the swapped scores are not symmetric. The score increase of a stronger player (with larger x_i) by defeating an underdog is less than its loss if it loses the game. Similarly, a player acquires more by defeating a stronger opponent than a weaker one.

There is a drawback of the definition in (1). Let us consider two players k and l. Player k loses all games it has played. In other words, $a_{kj} = 0$ for all $j = 1, \ldots, n$ and the kth row in the adjacency matrix $\mathbf{A}$ contains only zeroes. It is easy to see that its score x_k must be zero since all coefficients $\frac{a_{kj}}{a_{kj}+a_{jk}}$ are zero. On the other hand, player l wins all games against player k but loses all other games. Still, player l should have a zero score because either $a_{lj} = 0$ for $j \neq k$ or $x_k = 0$ and thus $x_l = 0$ by (1). This does not make much sense in practice [Newman, 2018]. In fact, being able to compete against other players is an affirmation for its skill and capability. Not many players have the chance of playing against top players in sports. If they do, it is very likely that they are participating in some top-notch tournaments which implies that their skills must be at a relatively high level. Moreover, every player should get some credit for being able to compete against other players. One should gain something even though a game is lost. As a remedy, we propose to generalize (1) as

$$x_i = \lambda \sum_{j \in \mathrm{Ne}(i)} \frac{a_{ij} + \alpha}{a_{ij} + a_{ji} + 2\alpha} x_j + \beta. \tag{2}$$

The newly added term β is the constant extra amount that every player receive, similar to the counterpart in the Katz centrality [Katz, 1953]. This term ensures that all players have a positive score, even though they lose all games they have played, and thus pass it along to others in the network. The new term α in the coefficient represents the gain from playing against other players.

Equation (2) can be written in the matrix form $\mathbf{x} = \lambda \widetilde{\mathbf{A}} \mathbf{x} + \beta \mathbf{1}$ where $\widetilde{\mathbf{A}}$ is an $n \times n$ matrix with elements $\widetilde{A}_{ij} = \frac{a_{ij}+\alpha}{a_{ij}+a_{ji}+2\alpha} = \frac{a_{ij}+\alpha}{n_{ij}+2\alpha}$ if $n_{ij} > 0$, and zero otherwise; and $\mathbf{1} = (1, 1, \ldots, 1)$ is an n-vector of ones. The solution is $\mathbf{x} = \beta(\mathbf{I} - \lambda \widetilde{\mathbf{A}})^{-1}\mathbf{1}$ conditioned on the fact the inverse exists. The overall coefficient β is not important since we typically only care about the relative magnitude (rank) of scores and a multiplication of an overall constant will not change the ranks. We can use $\beta = 1$ for simplicity. Additionally, in order for the inverse to exist, λ must be less than the reciprocal of the largest (most positive) eigenvalue of $\widetilde{\mathbf{A}}$ [Newman, 2018].

Some generalizations can be accommodated for more realistic situations. First of all, if a tie/draw is allowed as in chess and soccer games, a traditional way is to assign a half win to both players. If a weaker player draws a game against a stronger player, it is considered a "victory" for the underdog who then gains more from the score swapping than its opponent does. Additionally, different games may have distinct importance. For instance, winning a

Grand Slam final should bring more glory and honor than winning a qualifying match. a_{ij} needs to be replaced by w_{ij}, the total weight of winning games of player i against player j. Equation (2) becomes

$$x_i = \lambda \sum_{j \in \mathrm{Ne}(i)} \frac{w_{ij} + b_{ij}/2 + \alpha}{w_{ij} + a_{ji} + b_{ij} + 2\alpha} x_j + \beta.$$

where b_{ij} is the number of ties between player i and player j.

In the simulation study, we assume that there are 50 players in a multi-player game network. Their skill levels u_i ($i = 1, \ldots, 50$) are equally spaced on the interval of $[1, 5]$. Higher values of skill level indicate stronger players. The strongest player is player 1 with $u_1 = 5$ and the weakest player is player 50 with $u_{50} = 1$. The number of games n_{ij} completed between a pair of players i and j follows a Poisson distribution with mean 8. The probability that player i wins any game against player j is $p_{ij} = u_i/(u_i + u_j)$, and the results of all games are independent. The simulation is carried out 200 times. In each repetition, all players are ranked and we tabulate all 200 ranked lists and summarize using the heat map in Figure 2. The darkness of the grid represents the frequency of the pairs of true and estimated ranks.

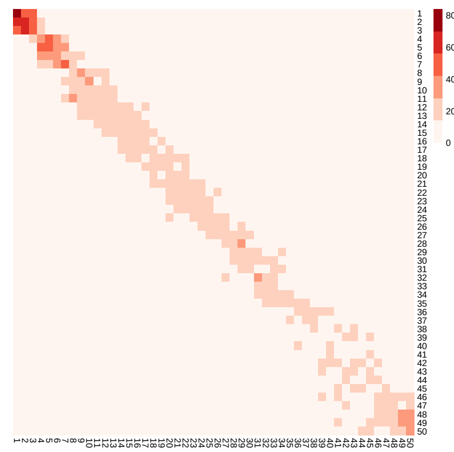

Figure 2: A heat map summary of 200 simulation repetitions. The horizontal axis is the true ranking of the players. The vertical axis is the estimated ranking.

A real data analysis is also performed on the records of ATP Tour in 2019 [Sackmann, 2022]. The results are compared to the merit-based method used by ATP. Other applications include the identification of disease-associated top (hub) genes in RNA sequencing-derived gene-gene interaction data (e.g., correlation or weights).

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