Dynamics of the Ising model over highly connected random graphs with arbitray degree distribution

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The theory of random graphs is of fundamental importance to describe systems where the structure of the underlying complex network affects its behaviour. One example are the Ising models, a set of discrete dynamical variables (± 1), or spins, whose state are determined by the interaction with their neighbours (specially by the coordination degree, the number of connections a spin performs), a feature completely dependent on the topology of the network at which they are defined upon. From a modelling perspective, Ising models provide a very general and simple tool with a wide range of applications, such as the study of phase transitions in statistical physics models [1], the mapping of NP problems in computer science [2] and the dynamical opinion formation on social networks [3]. In this work, we address the out-of-equilibrium dynamics of the ferromagnetic Ising model defined over highly connected simple random graphs with arbitrary degree distributions, capturing the effects of network structure on the macroscopic evolution that are not counted for when considering fully connected systems [4]. We provide the analytical solution for the discrete time evolution in the form of a closed set of dynamical equations that, for the long time limit, generalize the fixed point equations for the stationary states derived through equilibrium statistical mechanics in an early work [1]. In adittion, the generality of our formulation allows for the investigation of different noise distributions in the stochastic dynamics, that may be of interest when considering non-physical applications.

We consider a system with N Ising spins, defined upon a simple random graph whose microscopic structure is encoded by the degree sequence $(K_1, \cdots, K_N)$, where K_i is a random variable drawn from the probability distribution p_K that determines the number of connections the ith spin performs. Each connected pair of spins interacts via a constant ferromagnetic coupling energy $J > 0$. Starting from a random initial configuration, in the high connectivity limit, we show that the evolution of the macroscopic state m is given by

$$m_{t+1} = \int_0^\infty dg \nu(g) G(\beta J g u_t), \tag{1}$$

$$u_{t+1} = \int_0^\infty dg g \nu(g) G(\beta J g u_t), \tag{2}$$

where β is the inverse temperature parameter, $\nu(g)$ is the rescaled degree distribution

$$\nu(g) = \lim_{c\to\infty} \sum_{k\geq 0} p_k \delta\left(g - \frac{k}{c}\right) \tag{3}$$

and $G(x)$ is the activation function, associated with the stochastic noise distribution $\mu(\xi)$ through

$$G(x) = \int_{-x}^{x} d\xi \mu(\xi). \tag{4}$$

Regarding noise distributions, we consider the standart hyperbolic distribution μ_h, that recover thermal equilibrium results [5], and an algebraic form $\mu_a^{(\kappa)}$ that breaks detailed balance, rendering usual equilibrium statistics techniques unsuitable to the analysis of stationary states. Considering a negative binomial distribution p_K (with variance v^2), we are able to investigate the role of degree fluctuations on the system dynamics in terms of a single parameter α, associated with the rescaled distribution variance as

$$\lim_{c\to\infty} \frac{v^2}{c^2} = \frac{1}{\alpha}. \tag{5}$$

Therefore, in the limit of $\alpha \to \infty$ we recover the behaviour of a fully connected model, where the usual mean field theories fit perfectly, while the effects of degree fluctuations arises for finite α. We characterize the relaxation time as a function of α for different noise distributions, obtaining for the critical relaxation ($t >> 1$)

$$m_t^h \sim t^{-\frac{1}{2}}, \quad m_t^a \sim t^{-\frac{1}{2\kappa}}. \tag{6}$$

These results are presented on the left panel of figure (1). In the limit of large t (when $u_{t+1} = u_t, \forall t$), the set (1) and (2) is reduced to a fixed point equation for u that defines m, whose solution is presented in the right panel of figure (1). We can see the tendency towards the fully connected Ising ferromagnet (or the Currie-Weiss model, in the case of hyperbolic tangent activation [6]) as α increases. We also obtain the critical temperature T_c as a function of α, reassuring the result from [1], and go further to the critical exponents of the magnetization, obtaining for $T \to T_c$

$$m_h \sim C_h \left(\frac{T - T_c}{T_c}\right)^{\frac{1}{2}}, \quad m_a \sim C_a(\kappa) \left(\frac{T - T_c}{T_c}\right)^{\frac{1}{2\kappa}}. \tag{7}$$

Remarkably, the values of the critical equilibrium exponents are the same of the critical dynamical ones.

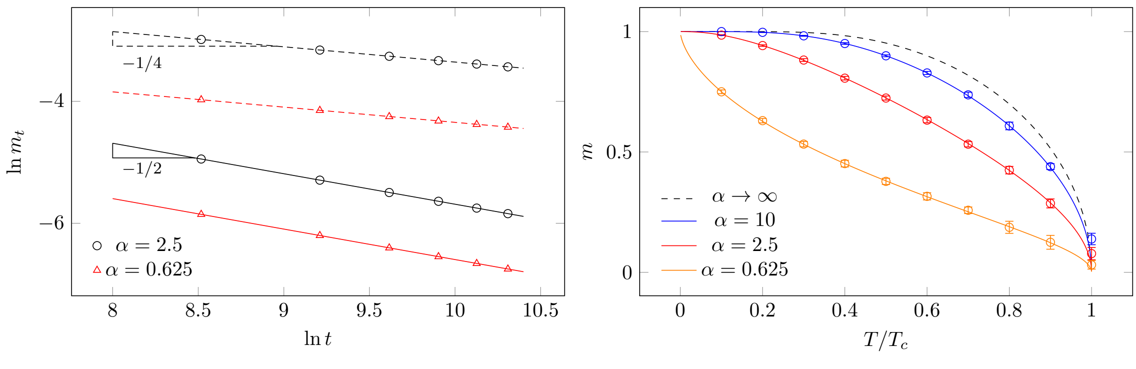

Fig. 1: The left panel presents the logarithmic relaxation for the algebraic distribution with index value $\kappa = 1$ (solid lines) and $\kappa = 2$ (dashed lines), with slope given by $-\frac{1}{2\kappa}$; the symbols are results of iteration of (1) and (2). The right panel presents a comparison between stationary limit of (1) and (2) (lines) and Monte Carlo simulations (symbols). Black dashed line denote the fully connected regime.

Overall, our work introduces a family of Ising models over random graphs that retain the effect of both topological structure and noise distribution whose non-equilibrium dynamics can be solved exactly, presenting insights on the network effect on the dynamics.

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