From subcritical behavior to a correlation-induced transition in rumor models

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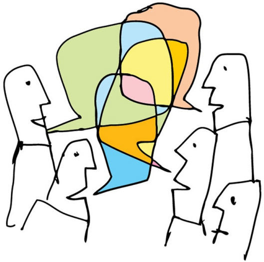
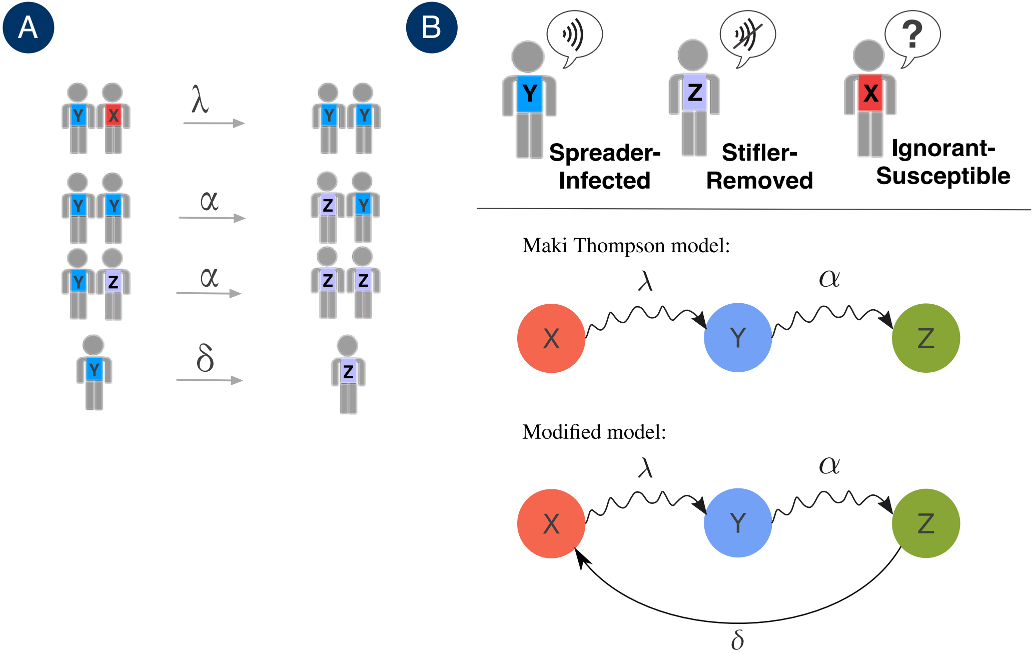
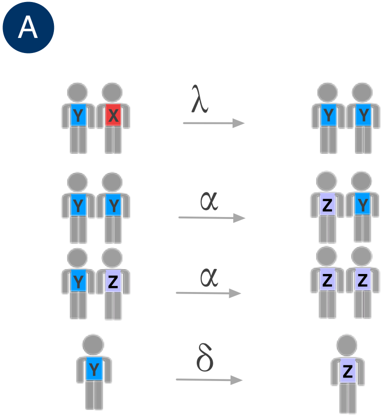
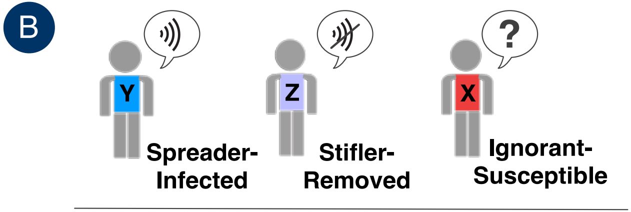


Image adapted from: https://pragmatics.indiana.edu/textbook/cap5.html


A
B
λ
α
α
δ
Y
X
Z
Spreader-
Infected
Stifler-
Removed
Ignorant-
Susceptible
Maki Thompson model:
Modified model:





$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\lambda\langle k\rangle i(t)s(t) \tag{10.2}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = +\lambda\langle k\rangle i(t)s(t) - \alpha\langle k\rangle s(t)[s(t) + r(t)] \tag{10.3}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha\langle k\rangle s(t)[s(t) + r(t)]. \tag{10.4}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}r_\infty}(1 - \mathrm{e}^{-\beta r_\infty})\right|_{r_\infty=0} > 1, \tag{10.10}$$

$$\frac{\lambda}{\alpha} > 0. \tag{10.11}$$

Barrat, A., Barthélemy, M., & Vespignani, A. (2008). Dynamical Processes on Complex Networks. Cambridge: Cambridge University Press.

PHASE TRANSITION:

- MT model has infinitely many absorbing states
- **The critical point is defined as the parameter that separates two scaling regimes:**
  - The final number of stiflers does not scale with the system size (zero in the thermodynamic limit);
  - The number of stiflers scales with the system size.

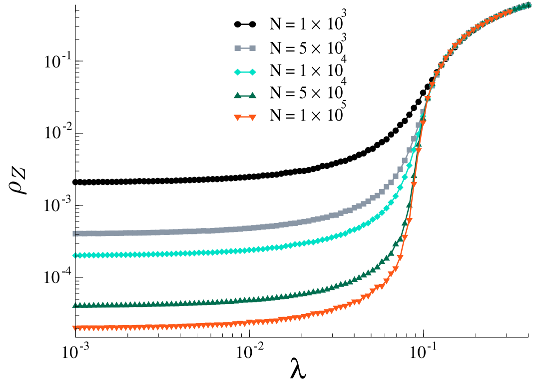


Fig. 2 Phase diagram for the standard MT model. Results for $\alpha = 1$ and different sizes on a random regular networks with $\langle k \rangle_k = 10$.

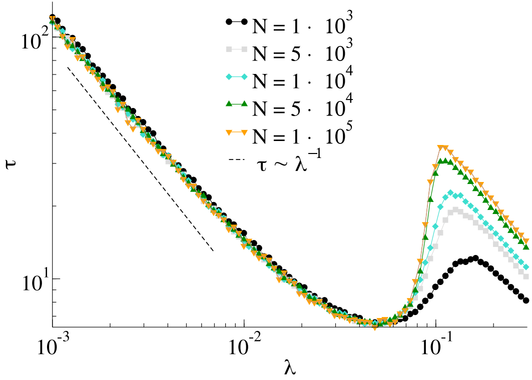


Fig. 3 Time to reach the absorbing state for the standard MT model. Results for $\alpha = 1$ and different sizes on random regular networks with $\langle k \rangle_k = 10$. The dashed line follows $\tau_f \sim \lambda^{-1}$.

CRITICAL POINT ($\alpha \gg \delta$)

$$q_k(i) = \begin{cases} \frac{(k-i)\lambda}{i\alpha+(k-i)\lambda}\left[i\frac{(i+1)\alpha}{(i+1)\alpha+(k-i-1)\lambda}+q_k(i+1)\right] & \text{if } i<k \\ 0 & \text{otherwise} \end{cases}$$

$$q(1) = \langle q_k(1)\rangle_k > 1$$

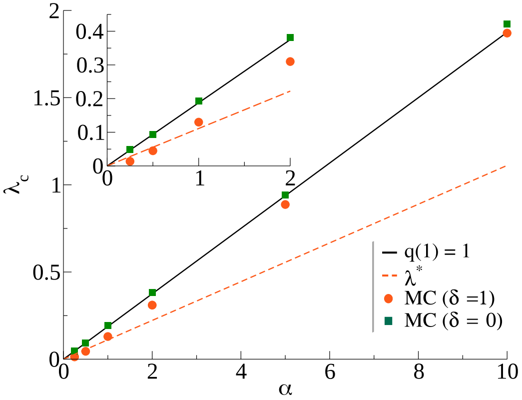


Fig. 4 Comparison between analytical and Monte Carlo critical point estimations (λ_c). Results for random regular networks with $\langle k\rangle_k = 10$ and $\delta = 1$ and $N = 10^6$. The continuous line expresses the value of λ_c obtained as a solution of $q(1) = 1$, from Eq. (5). In contrast, the dashed line represents the naive approximation that accounts only for the probability that the next event is spreading or stifling. In the inset we present the comparison for the low α regime.

Power-law behavior: $P(k) \sim k^{-\gamma}$

- $2.0 < \gamma < 3.0$: vanishing;
- $\gamma > 3.0$: non-null;
- Robust for a range of α.

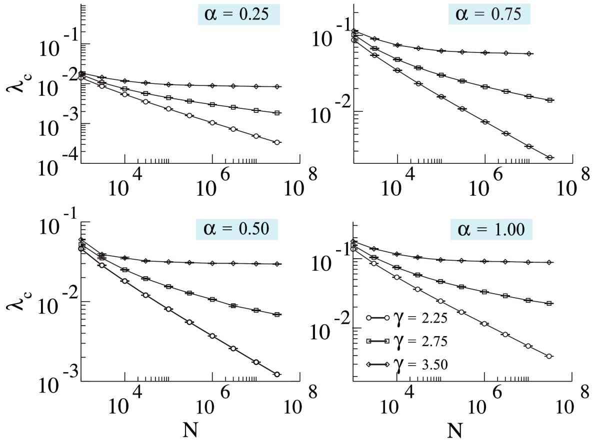


Fig. 5 Critical point estimations of uncorrelated power-law networks. We plot λ_c as a function of N and for different values of γ and α, considering $\delta = 1$.

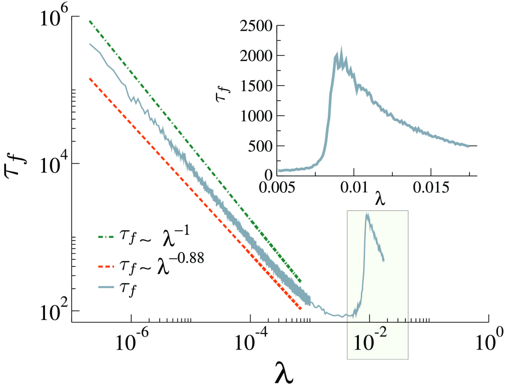


Fig. 6 Lifespan as a function of the spreading rate λ. Results for $\delta = 1$, $\alpha = 0.5$ on an uncorrelated power-law network with $P(k) \sim k^{-\gamma}$ with $\gamma = 2.25$, $N = 10^6$. In the main panel, we show a wide range of λ, emphasizing the sub-critical behavior, while in the inset we show the peak that suggests a continuous phase transition. The blue curve (dot dashed line) follows $\tau_f \sim \lambda^{-1}$ and the orange curve (dashed line) follows $\tau_f \sim \lambda^{-0.88}$, obtained from a fitting of the lifespan obtained using Monte Carlo simulations (the gray curve).

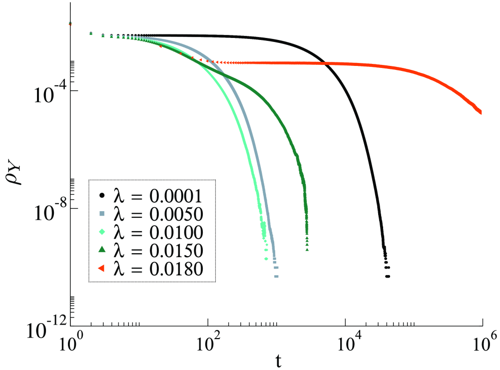
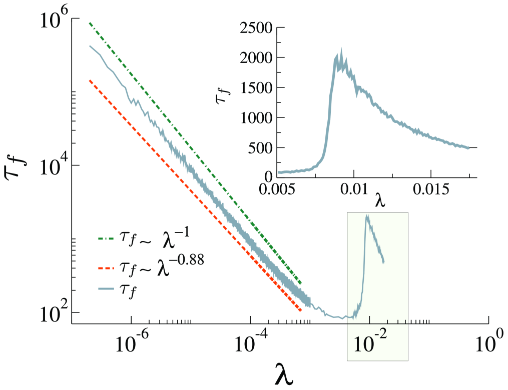


Fig. 7 Temporal behavior of the density of spreaders for an uncorrelated power-law network. Result for a network with $N = 10^6$, $\gamma = 2.25$, $\alpha = 0.5$ and $\delta = 1$ in the regime $\lambda \ll \delta$ for values of λ near the critical point $\lambda_c \approx 0.015$.



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Subcritical behavior

$$P_1 = 1 \qquad \langle \tau_1 \rangle = \frac{1}{\langle k \rangle_k \lambda}$$

$$P_2 = \frac{2\alpha}{2\alpha + 2\lambda(\langle k \rangle_k - 1)} \qquad \langle \tau_2 \rangle = \frac{1}{2\alpha + 2\lambda(\langle k \rangle_k - 1)}$$

$$P_3 = \frac{\alpha}{\alpha + \lambda(\langle k \rangle_k - 1) + \delta} \qquad \langle \tau_3 \rangle = \frac{1}{\alpha + \lambda(\langle k \rangle_k - 1)}$$

$$P_4 = P_5 = 1 \qquad \langle \tau_4 \rangle = \langle \tau_5 \rangle = \frac{1}{\delta}.$$

Approximate behavior

$$T^* = \left(\frac{(\alpha + \delta)\delta}{\langle k \rangle_k \alpha^2} + \frac{\alpha}{\langle k \rangle_k (\alpha + \delta)} \right) \lambda^{-1}$$

$$\begin{cases} \frac{dx_i}{dt} = & \delta z_i - \lambda \sum_{k=1}^{N} \mathbf{A}_{ki} x_i y_k \\ \frac{dy_i}{dt} = & \lambda \sum_{k=1}^{N} \mathbf{A}_{ki} x_i y_k - \alpha \sum_{k=1}^{N} \mathbf{A}_{ki} y_i \left(y_k + z_k\right) \\ \frac{dz_i}{dt} = & -\delta z_i + \alpha \sum_{k=1}^{N} \mathbf{A}_{ki} y_i \left(y_k + z_k\right). \end{cases}$$

$$\begin{cases} \delta z_i^{(1)} \epsilon^k & = \tilde{\lambda} \epsilon^m \sum_{k=1}^{N} \mathbf{A}_{ki} x_i^{(1)} y_k^{(1)} \epsilon^c \\ \delta z_i^{(1)} \epsilon^k & = \tilde{\alpha} \epsilon^n \sum_{k=1}^{N} \mathbf{A}_{ki} y_i^{(1)} \epsilon^c \left(y_k^{(1)} \epsilon^c + z_k^{(1)} \epsilon^k \right). \end{cases}$$

Asymptotic analysis

$k = n + c + \min(c, k)$ and $k = m + c$, establishing three different possible regimes:

- $0 < c < k$, where $c = m - n$ and $k = 2m - n$;
  $y_i^{(1)} = \frac{\tilde{\lambda}}{\tilde{\alpha}}$
- $c = k$, where $m = 0$ and $n = -k$;
  $y_i \sim \frac{\lambda}{\alpha(\lambda \Lambda_{\max} + 1)}$
- $0 < k < c$, where $c = -n$ and $k = m - n$.
  $y_i \sim \frac{1}{\alpha \Lambda_{\max}}$, $\Lambda_{\max} \approx \langle k \rangle_k$

Acknowledgments

Thank you!

CENTAI

- Ferraz de Arruda, G., Jeub, L.G.S., Mata, A.S. et al. **From subcritical behavior to a correlation-induced transition in rumor models.** Nat Commun 13, 3049 (2022). https://doi.org/*10.1038/s41467-022-30683-z*
- *http://guifarruda.gitlab.io/*