Analytic solution for the spectral density and localization properties of complex networks

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Introduction

- The leading eigenpair of the adjacency random matrix governs:
  - spreading of diseases;
  - synchronization transition;
  - stability of complex systems.
- Analytical solutions only for the **homogeneous case**;
  - regular random graphs;
  - high-connectivity limit.
- How do we extract analytic information from **heterogeneous random networks**?
- Study the spectral and localization properties of heterogeneous random networks in the **high connectivity limit**.

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The problem setup I

- Simple and undirected random graph with N nodes and average degree c;
- Network topology specified by the weighted adjacency random matrix $\mathbf{A}$ with entries given by $A_{ij} = c_{ij} J_{ij}$;
- $c_{ij} = 1$ if i and j are connected and $c_{ij} = 0$ otherwise;
- J_{ij} i.i.d random variable distributed as p_J with mean zero and standard deviation $J_1/\sqrt{c}$;
- $\mathbf{A}$ is generated according to the configuration model of networks.

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The main equations I

Spectral density

$$\rho(\lambda) = \lim_{N \to \infty} \frac{1}{N} \sum_{\mu=1}^{N} \delta(\lambda - \lambda_\mu)$$

LDOS

$$\rho_i(\lambda) = \sum_{\mu=1}^{N} |v_{\mu,i}|^2 \delta(\lambda - \lambda_\mu)$$

IPR

$$Y_\mu = \sum_{i=1}^{N} (v_{\mu,i})^4$$

Eigenvalue-dependent IPR

$$I(\lambda) = \lim_{N \to \infty} \frac{\sum_{\mu=1}^{N} \delta(\lambda - \lambda_\mu) Y_\mu}{\sum_{\mu=1}^{N} \delta(\lambda - \lambda_\mu)}$$

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The main equations II

Resolvent matrix

$$G(z) = (Iz - A)^{-1}, \;\; z = \lambda - i\epsilon$$

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$$\rho_\epsilon(\lambda) = \frac{1}{\pi} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathrm{Im} G_{ii}(z)$$

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Cavity method

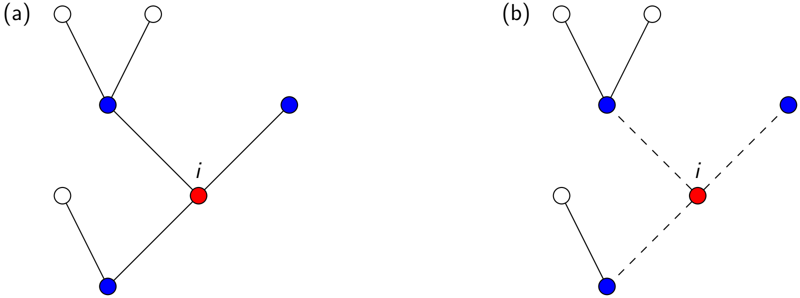

Figure: Locally tree-like structure.

The high connectivity limit

Cavity equations

$$G_{ii}(z) = \frac{1}{z - \sum_{j \in \partial_i} J_{ij}^2 G_{jj}^{(i)}(z)} \quad (i = 1, \ldots, N)$$

The symbol ∂_i denotes the set of nodes adjacent to i.

$$G_{jj}^{(i)}(z) = \frac{1}{z - \sum_{\ell \in \partial_j \setminus i} J_{j\ell}^2 G_{\ell\ell}^{(j)}(z)} \quad i \in \partial_j$$

Law of large numbers
$(1 \ll c \ll N)$

$$\sum_{j \in \partial_i} J_{ij}^2 G_{jj}^{(i)}(z) \xrightarrow{c \to \infty} \frac{k_i}{c} J_1^2 \langle G \rangle$$

pdf of the re-scaled degrees

$$\nu(\kappa) = \lim_{c \to \infty} \sum_{k=0}^{\infty} p_k \delta\left(\kappa - \frac{k}{c}\right)$$

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Equations for the observables

Average resolvent on the cavity graph

$$\langle G\rangle = \int_0^\infty \frac{d\kappa\,\nu(\kappa)\kappa}{z - \kappa J_1^2 \langle G\rangle}$$

Spectral density

$$\rho_\epsilon(\lambda) = \frac{1}{\pi}\mathrm{Im}\left[\int_0^\infty \frac{d\kappa\,\nu(\kappa)}{z - \kappa J_1^2 \langle G\rangle}\right]$$

Inverse participation ratio

$$I_\epsilon(\lambda) = \frac{\epsilon}{\pi\rho_\epsilon(\lambda)}\int_0^\infty \frac{d\kappa\,\nu(\kappa)}{|z - \kappa J_1^2 \langle G\rangle|^2}$$

pdf of the LDOS at $z = 0$

$$P_0(y) = \frac{1}{J_1 y^2}\nu\left(\frac{1}{J_1 y}\right), \;\; y_i = \mathrm{Im} G_{ii}$$

$z = 0 \Rightarrow \langle G\rangle = \frac{i}{J_1}$ for any $\nu(\kappa)$.

Equations for the observables

Average resolvent on the cavity graph

$$hG\,i = \int_0^{1} \frac{d\kappa\,\nu(\kappa)\kappa}{z \quad \kappa J_1^2 hG\,i}$$

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Negative binomial degree distribution

- Results for random graphs with a **negative binomial degree distribution**
- The pdf of the re-scaled degrees stands as

$$\nu_b(\kappa) = \frac{\alpha^\alpha \kappa^{\alpha-1} e^{-\alpha\kappa}}{\Gamma(\alpha)}$$

- For $c \to \infty$, the relative variance of $p_k^{(b)}$ reads

$$\lim_{c \to \infty} \frac{\sigma_b^2}{c^2} = \frac{1}{\alpha}$$

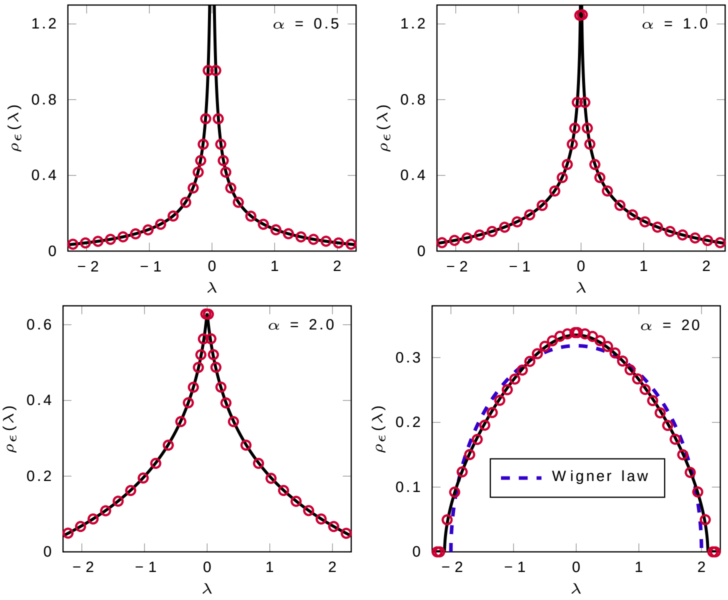
The spectral density

- α controls the heterogeneity
- α ! 1 =) homogeneous network
- α ! 0 strongly heterogeneous network

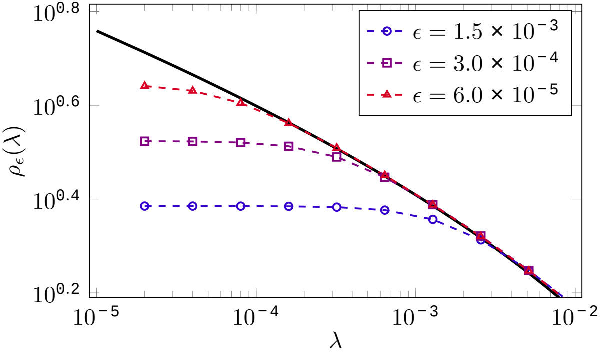
$$J_1^2 hG\,i^2 = \left(\frac{\alpha z}{J_1^2 hG\,i}\right)^{\alpha} \exp\left(\frac{\alpha z}{J_1^2 hG\,i}\right)\Gamma\left(1 \quad \alpha, \frac{\alpha z}{J_1^2 hG\,i}\right) \quad 1$$

$$\rho_\epsilon(\lambda) = \frac{1}{\pi}\mathrm{Im}\left[\frac{J_1^2 hG\,i^2 + 1}{z}\right]$$

The spectral density

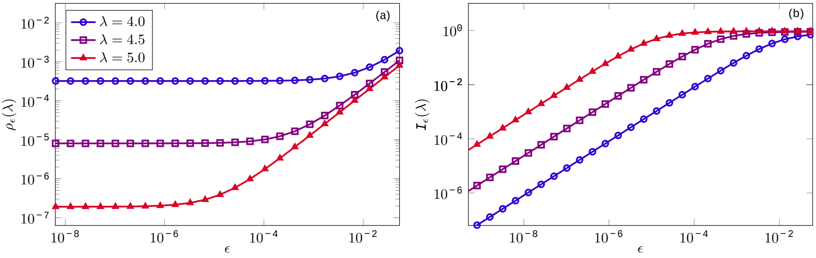


The spectral density for $|\lambda| \to 0$



$0 < \alpha < 1$	$\alpha = 1$	$\alpha > 1$
$\rho(\lambda) \sim |\lambda|^{\alpha-1}$	$\rho(\lambda) \sim \log|\lambda|$	$\rho(\lambda) = \frac{1}{\pi J_1}\frac{\alpha}{\alpha - 1}$

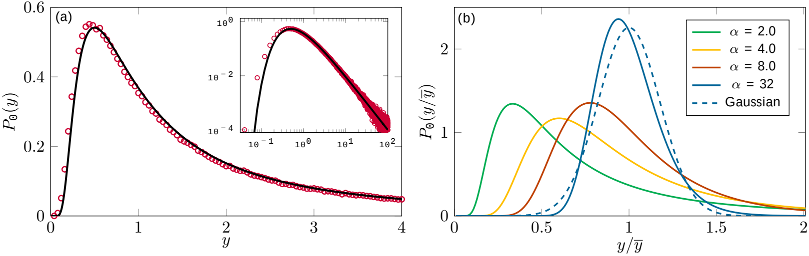
The IPR



For $\epsilon \to 0^+$

- $\rho_\epsilon(\lambda)$ converges to a finite value;
- $I_\epsilon(\lambda)$ vanishes as $I_\epsilon(\lambda) \quad \epsilon$;
- The same behavior holds for other values of α.

The distribution of the LDOS



$$P_0(y) = \frac{\alpha^\alpha}{\Gamma(\alpha) J_1^\alpha} \frac{e^{\;\frac{\alpha}{J_1 y}}}{y^{\alpha+1}}$$

$$\overline{y^q} = \frac{\alpha^q \Gamma(\alpha \quad q)}{J_1^q \Gamma(\alpha)}$$

Conclusion

- Spectral properties of heterogeneous random networks with **finite c do not have analytical form**;
- $c \to \infty$, **analytical solutions** for the adjacency matrix observables of **heterogeneous networks**;
- Negative binomial degree distribution:
  - Heterogeneity controlled by a single parameter α;
  - Singularity on the spectral density for $\alpha \leq 1$;
  - For $\lambda \neq 0$, IPR vanishes proportional to ϵ;
  - For $\lambda = 0$, $P_0(y)$ regular shape with a power-law tail;
  - Extended eigenvectors in the entire spectrum.

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Thank you!