

Analytic solution for the spectral density and localization properties of complex networks

Jeferson Dias da Silva and Fernando Lucas Metz

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Introduction

- ▶ The leading eigenpair of the adjacency random matrix governs:
 - ▶ spreading of diseases;
 - ▶ synchronization transition;
 - ▶ stability of complex systems.
- ▶ Analytical solutions only for the **homogeneous case**;
 - ▶ regular random graphs;
 - ▶ high-connectivity limit.
- ▶ How do we extract analytic information from **heterogeneous random networks**?
- ▶ Study the spectral and localization properties of heterogeneous random networks in the **high connectivity limit**.

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The problem setup I

- ▶ Simple and undirected random graph with N nodes and average degree c ;
- ▶ Network topology specified by the weighted adjacency random matrix \mathbf{A} with entries given by $A_{ij} = c_{ij}J_{ij}$;
- ▶ $c_{ij} = 1$ if i and j are connected and $c_{ij} = 0$ otherwise;
- ▶ J_{ij} i.i.d random variable distributed as p_J with mean zero and standard deviation J_1/\sqrt{c} ;
- ▶ \mathbf{A} is generated according to the configuration model of networks.

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The main equations I

Spectral density

$$\rho(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\mu=1}^N \delta(\lambda - \lambda_{\mu})$$

LDOS

$$\rho_i(\lambda) = \sum_{\mu=1}^N |v_{\mu,i}|^2 \delta(\lambda - \lambda_{\mu})$$

IPR

$$Y_{\mu} = \sum_{i=1}^N (v_{\mu,i})^4$$

Eigenvalue-dependent IPR

$$\mathcal{I}(\lambda) = \lim_{N \rightarrow \infty} \frac{\sum_{\mu=1}^N \delta(\lambda - \lambda_{\mu}) Y_{\mu}}{\sum_{\mu=1}^N \delta(\lambda - \lambda_{\mu})}$$

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The main equations II

Resolvent matrix

$$\mathbf{G}(z) = (\mathbf{I}z - \mathbf{A})^{-1}, \quad z = \lambda - i\epsilon$$

Spectral density

$$\rho_\epsilon(\lambda) = \frac{1}{\pi} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \text{Im} G_{ii}(z)$$

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Cavity method

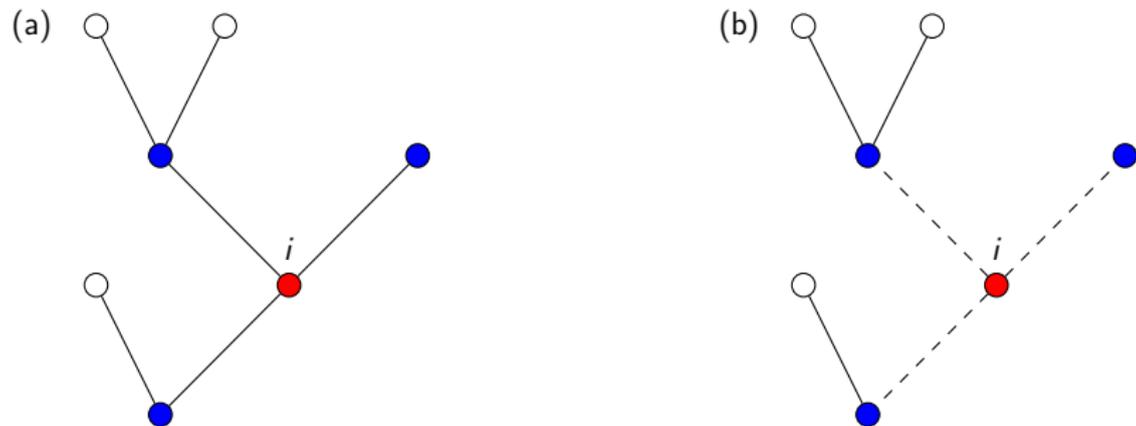


Figure: Locally tree-like structure.

The high connectivity limit

Cavity equations

$$G_{ii}(z) = \frac{1}{z - \sum_{j \in \partial_i} J_{ij}^2 G_{jj}^{(i)}(z)} \quad (i = 1, \dots, N)$$

The symbol ∂_i denotes the set of nodes adjacent to i .

$$G_{jj}^{(i)}(z) = \frac{1}{z - \sum_{\ell \in \partial_j \setminus i} J_{j\ell}^2 G_{\ell\ell}^{(j)}(z)} \quad i \in \partial_j$$

Law of large numbers
($1 \ll c \ll N$)

$$\sum_{j \in \partial_i} J_{ij}^2 G_{jj}^{(i)}(z) \xrightarrow{c \rightarrow \infty} \frac{k_i}{c} J_1^2 \langle G \rangle$$

pdf of the re-scaled degrees

$$\nu(\kappa) = \lim_{c \rightarrow \infty} \sum_{k=0}^{\infty} p_k \delta\left(\kappa - \frac{k}{c}\right)$$

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Equations for the observables

Average resolvent on the cavity graph

$$\langle G \rangle = \int_0^\infty \frac{d\kappa \nu(\kappa) \kappa}{z - \kappa J_1^2 \langle G \rangle}$$

Spectral density

$$\rho_\epsilon(\lambda) = \frac{1}{\pi} \text{Im} \left[\int_0^\infty \frac{d\kappa \nu(\kappa)}{z - \kappa J_1^2 \langle G \rangle} \right]$$

Inverse participation ratio

$$\mathcal{I}_\epsilon(\lambda) = \frac{\epsilon}{\pi \rho_\epsilon(\lambda)} \int_0^\infty \frac{d\kappa \nu(\kappa)}{|z - \kappa J_1^2 \langle G \rangle|^2}$$

pdf of the LDOS at $z = 0$

$$P_0(y) = \frac{1}{J_1 y^2} \nu \left(\frac{1}{J_1 y} \right), \quad y_i = \text{Im} G_{ii}$$

$$z = 0 \implies \langle G \rangle = \frac{i}{J_1} \text{ for any } \nu(\kappa).$$

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Negative binomial degree distribution

- ▶ Results for random graphs with a **negative binomial degree distribution**
- ▶ The pdf of the re-scaled degrees stands as

$$\nu_b(\kappa) = \frac{\alpha^\alpha \kappa^{\alpha-1} e^{-\alpha\kappa}}{\Gamma(\alpha)}$$

- ▶ For $c \rightarrow \infty$, the relative variance of $p_k^{(b)}$ reads

$$\lim_{c \rightarrow \infty} \frac{\sigma_b^2}{c^2} = \frac{1}{\alpha}$$

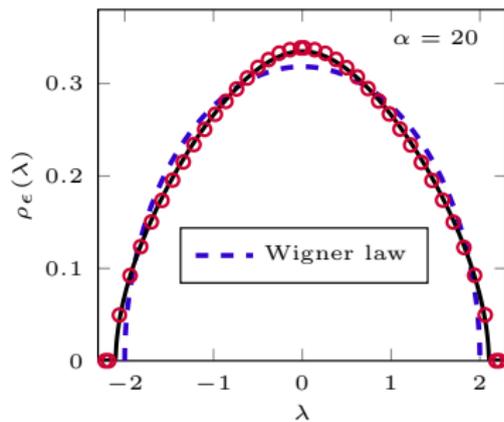
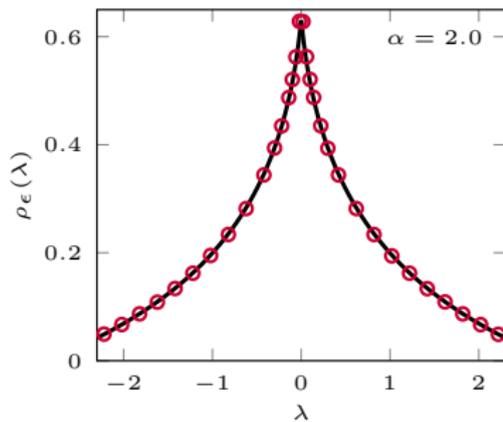
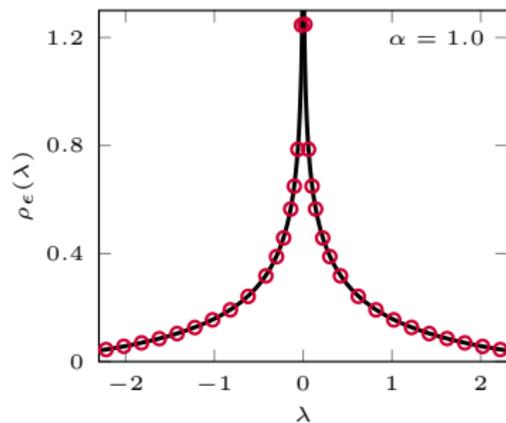
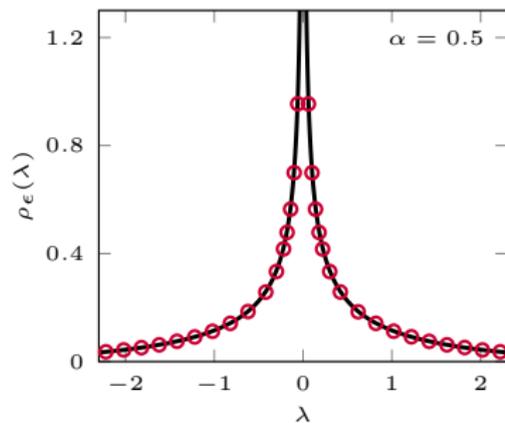
The spectral density

- ▶ α controls the heterogeneity
- ▶ $\alpha \rightarrow \infty \implies$ homogeneous network
- ▶ $\alpha \rightarrow 0$ strongly heterogeneous network

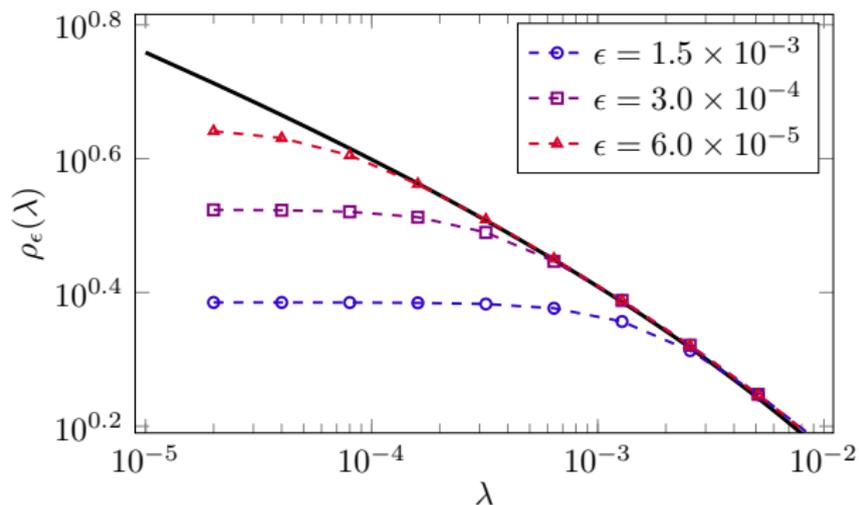
$$J_1^2 \langle G \rangle^2 = \left(\frac{-\alpha z}{J_1^2 \langle G \rangle} \right)^\alpha \exp \left(\frac{-\alpha z}{J_1^2 \langle G \rangle} \right) \Gamma \left(1 - \alpha, \frac{-\alpha z}{J_1^2 \langle G \rangle} \right) - 1$$

$$\rho_\epsilon(\lambda) = \frac{1}{\pi} \operatorname{Im} \left[\frac{J_1^2 \langle G \rangle^2 + 1}{z} \right]$$

The spectral density



The spectral density for $|\lambda| \rightarrow 0$



$$0 < \alpha < 1$$

$$\rho(\lambda) \sim |\lambda|^{\alpha-1}$$

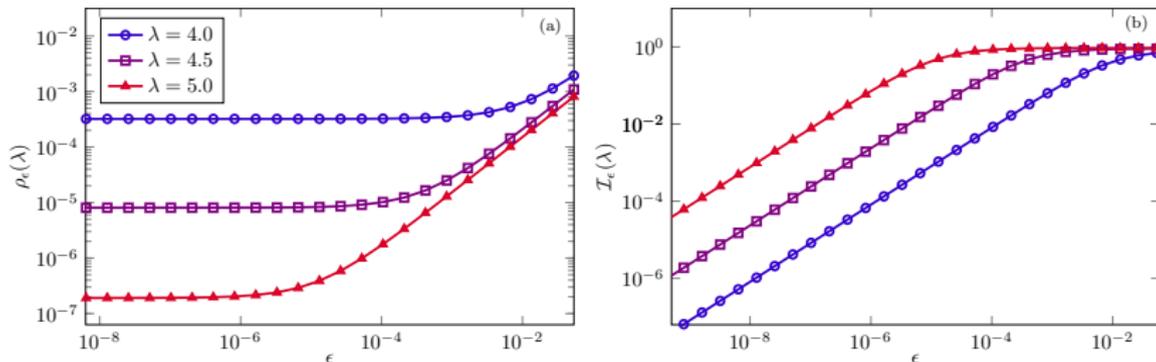
$$\alpha = 1$$

$$\rho(\lambda) \sim \log |\lambda|$$

$$\alpha > 1$$

$$\rho(\lambda) = \frac{1}{\pi J_1} \frac{\alpha}{\alpha - 1}$$

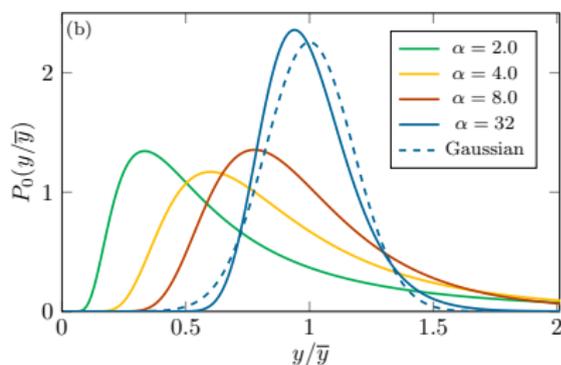
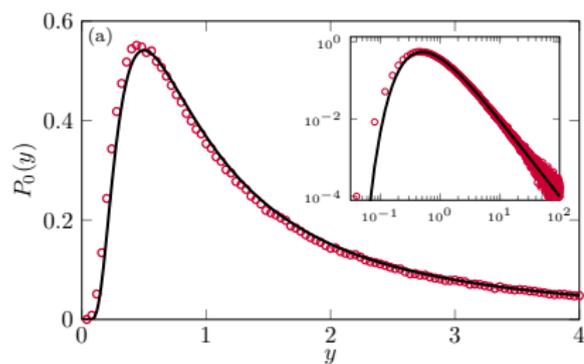
The IPR



For $\epsilon \rightarrow 0^+$

- ▶ $\rho_\epsilon(\lambda)$ converges to a finite value;
- ▶ $\mathcal{I}_\epsilon(\lambda)$ vanishes as $\mathcal{I}_\epsilon(\lambda) \sim \epsilon$;
- ▶ The same behavior holds for other values of α .

The distribution of the LDOS



$$P_0(y) = \frac{\alpha^\alpha}{\Gamma(\alpha) J_1^\alpha} \frac{e^{-\frac{\alpha}{J_1 y}}}{y^{\alpha+1}}$$

$$\overline{y^q} = \frac{\alpha^q \Gamma(\alpha - q)}{J_1^q \Gamma(\alpha)}$$

Conclusion

- ▶ Spectral properties of heterogeneous random networks with **finite c do not have analytical form**;
- ▶ $c \rightarrow \infty$, **analytical solutions** for the adjacency matrix observables of **heterogeneous networks**;
- ▶ Negative binomial degree distribution:
 - ▶ Heterogeneity controlled by a single parameter α ;
 - ▶ Singularity on the spectral density for $\alpha \leq 1$;
 - ▶ For $\lambda \neq 0$, IPR vanishes proportional to ϵ ;
 - ▶ For $\lambda = 0$, $P_0(y)$ regular shape with a power-law tail;
 - ▶ Extended eigenvectors in the entire spectrum.

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Thank you!