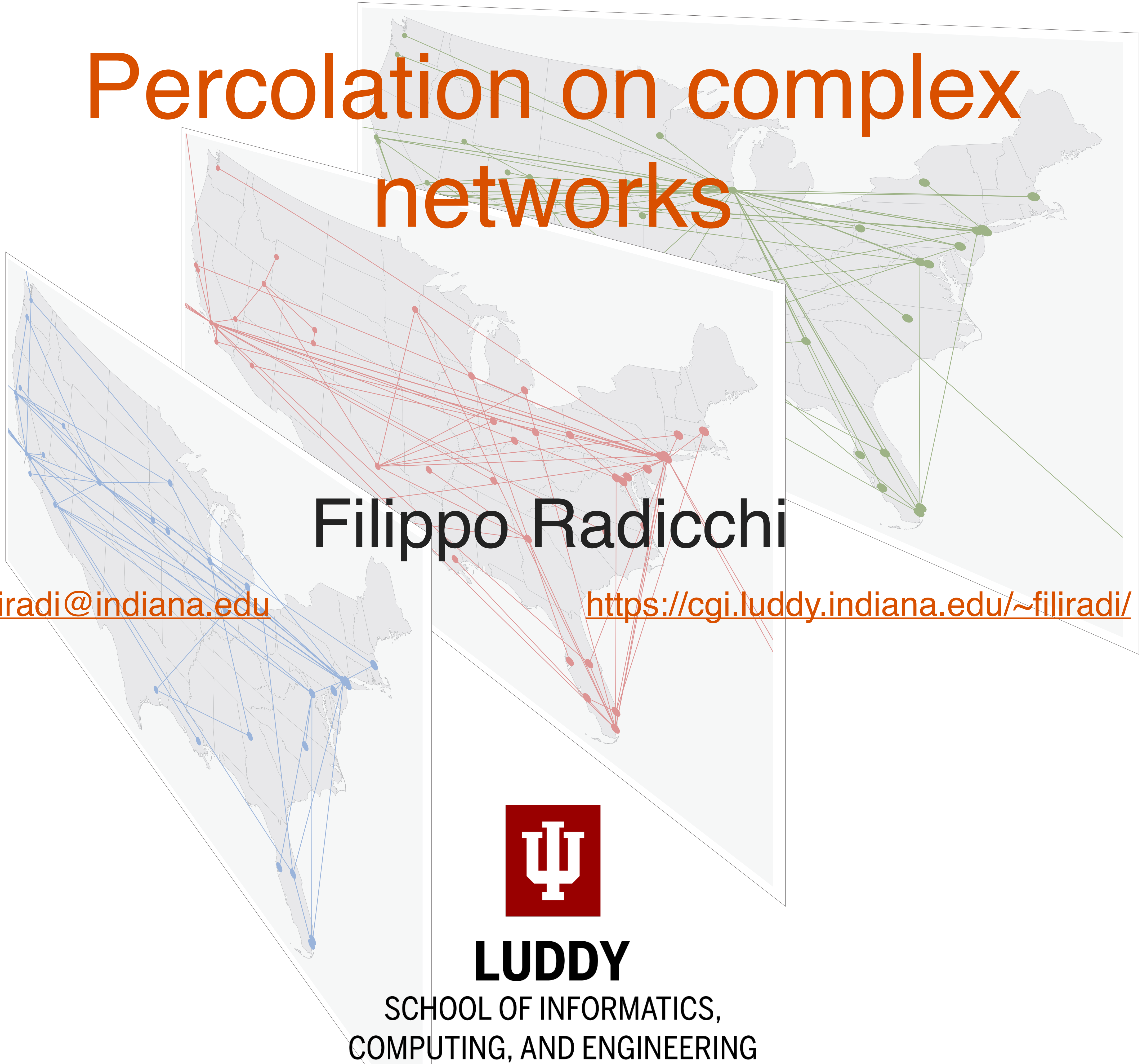
Percolation on complex networks



Filippo Radicchi

filiradi@indiana.edu

https://cgi.luddy.indiana.edu/~filiradi/

LUDDY
SCHOOL OF INFORMATICS, COMPUTING, AND ENGINEERING

my academic history

2003
Rome

2007
Bremen

2007-2010
Torino

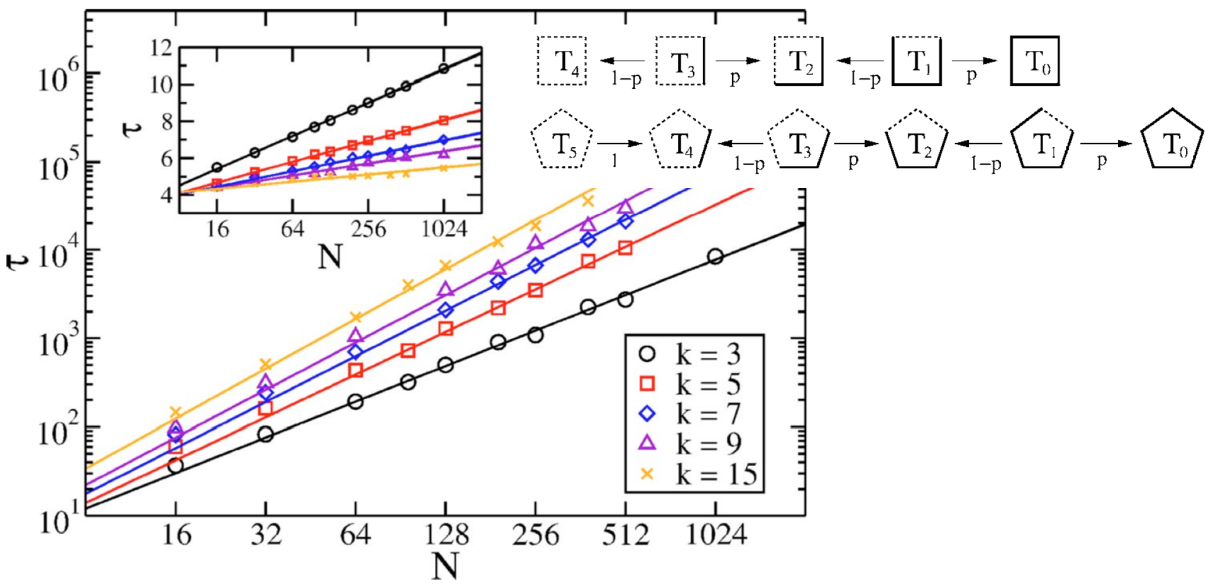
2010-2011
Chicago

2012-2013
Tarragona

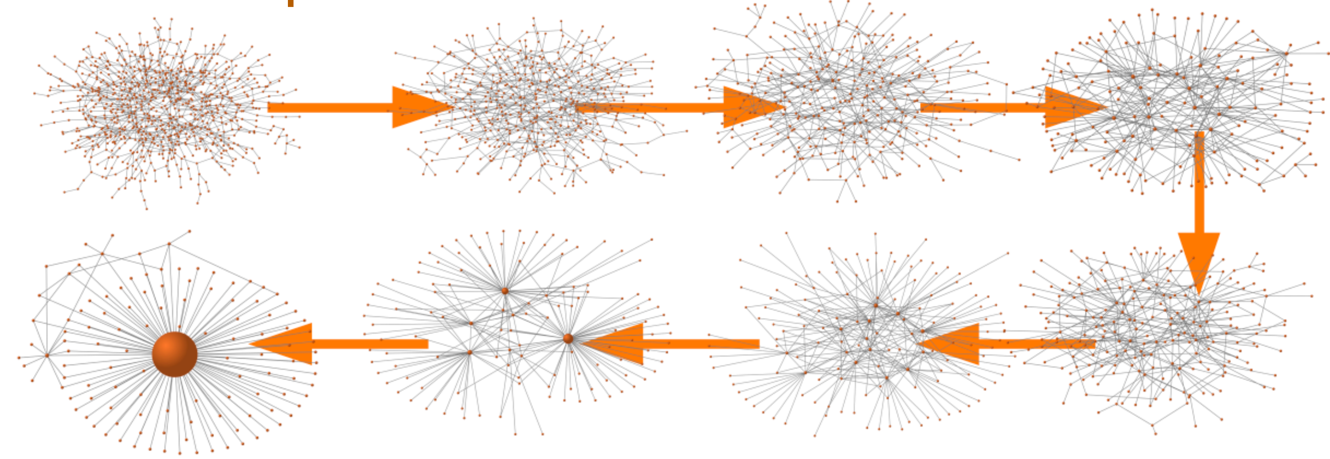
now @ IU

Statistical physics of complex networks

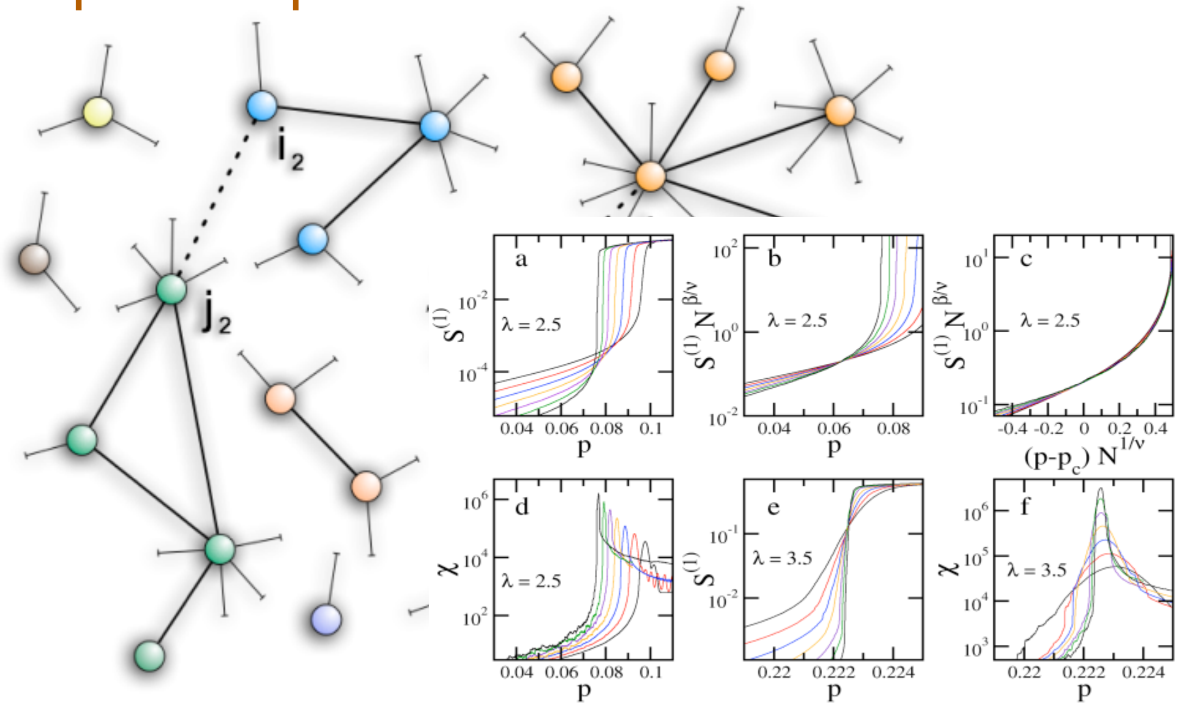
Spin dynamics in graphs



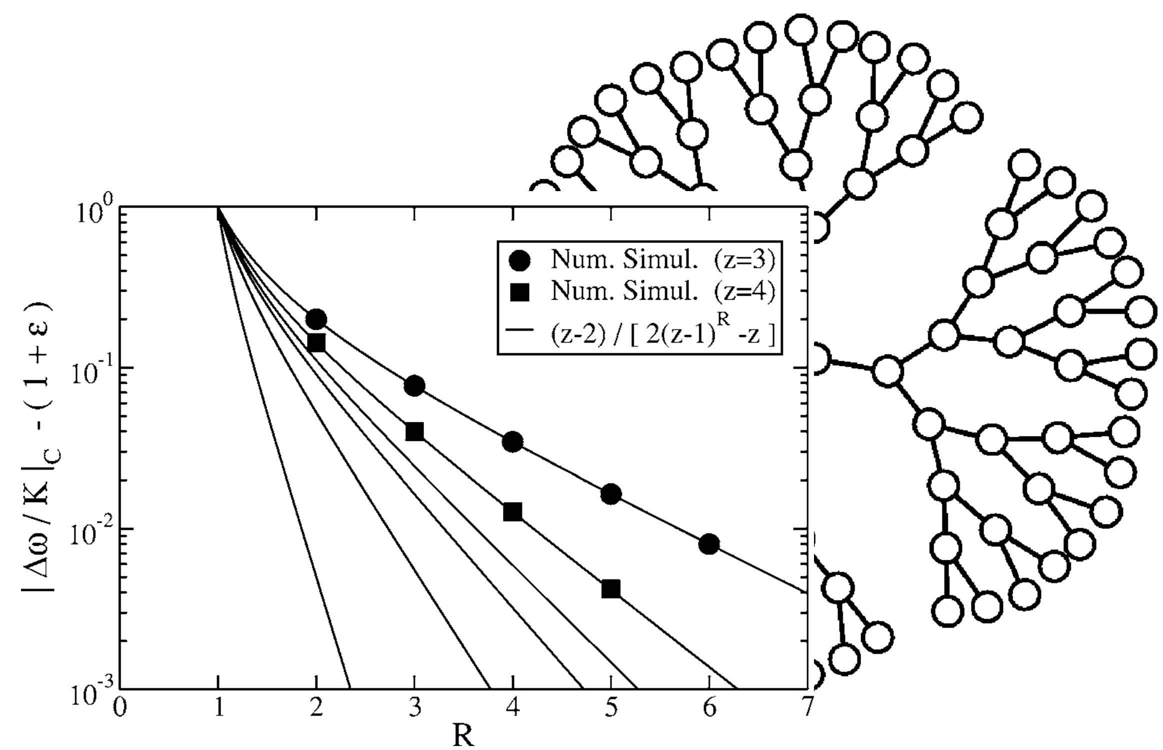
Complex networks renormalization flows



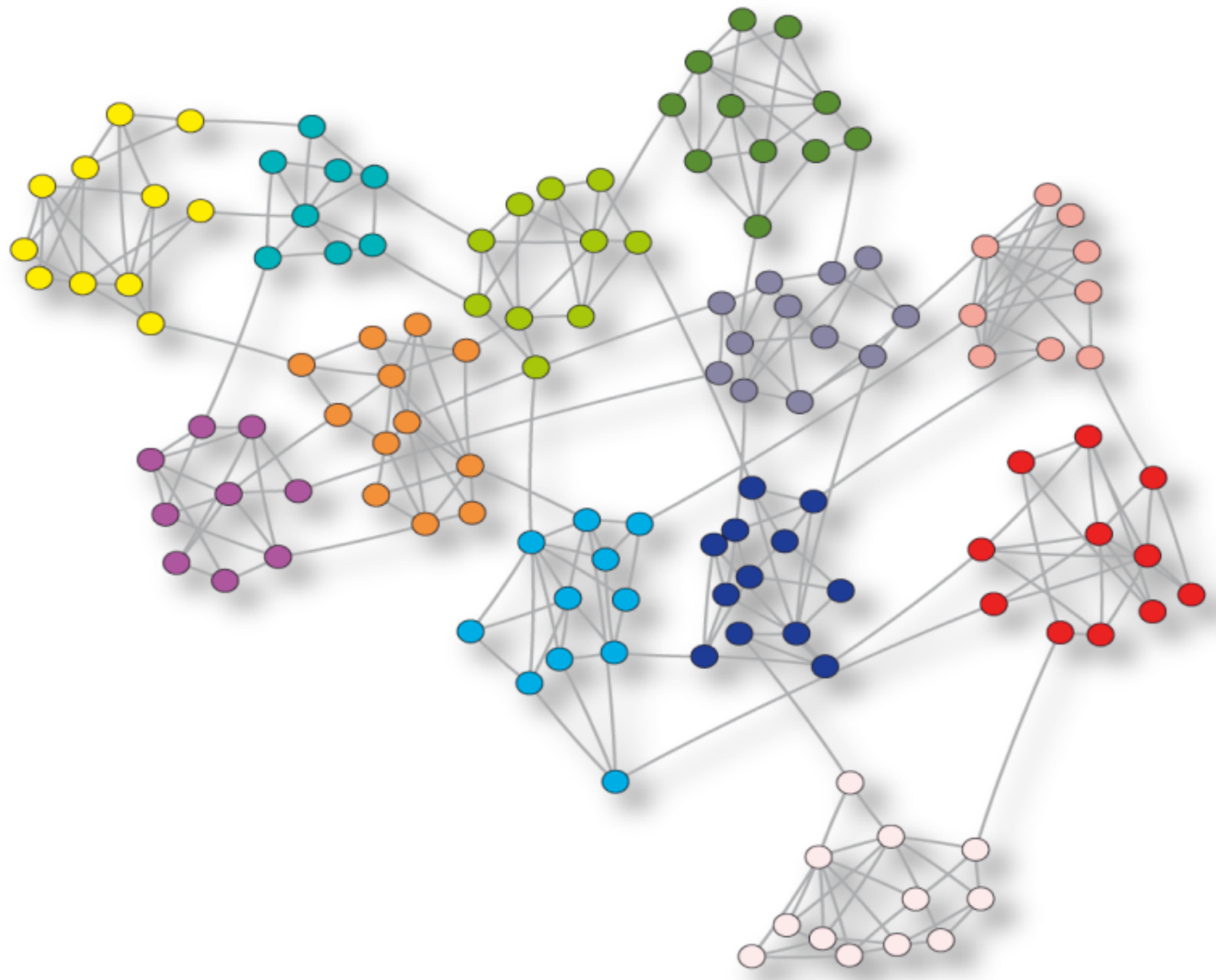
Explosive percolation in scale-free networks



Synchronization in networks



Community structure in networks



Citation networks
papers dealing with similar topics

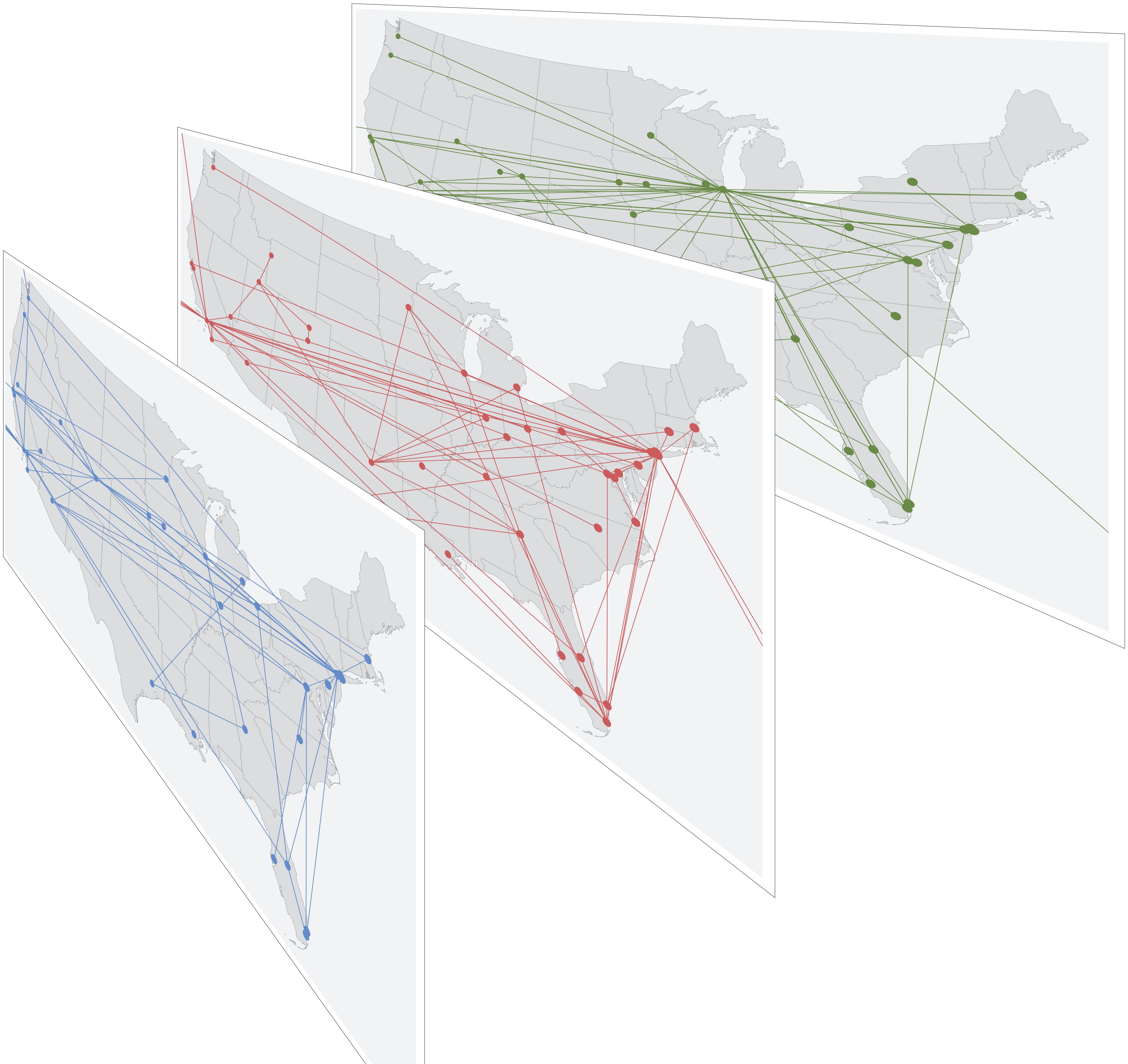
Protein interaction networks
proteins with similar biological functions

Social networks
people sharing common interests

Main contributions

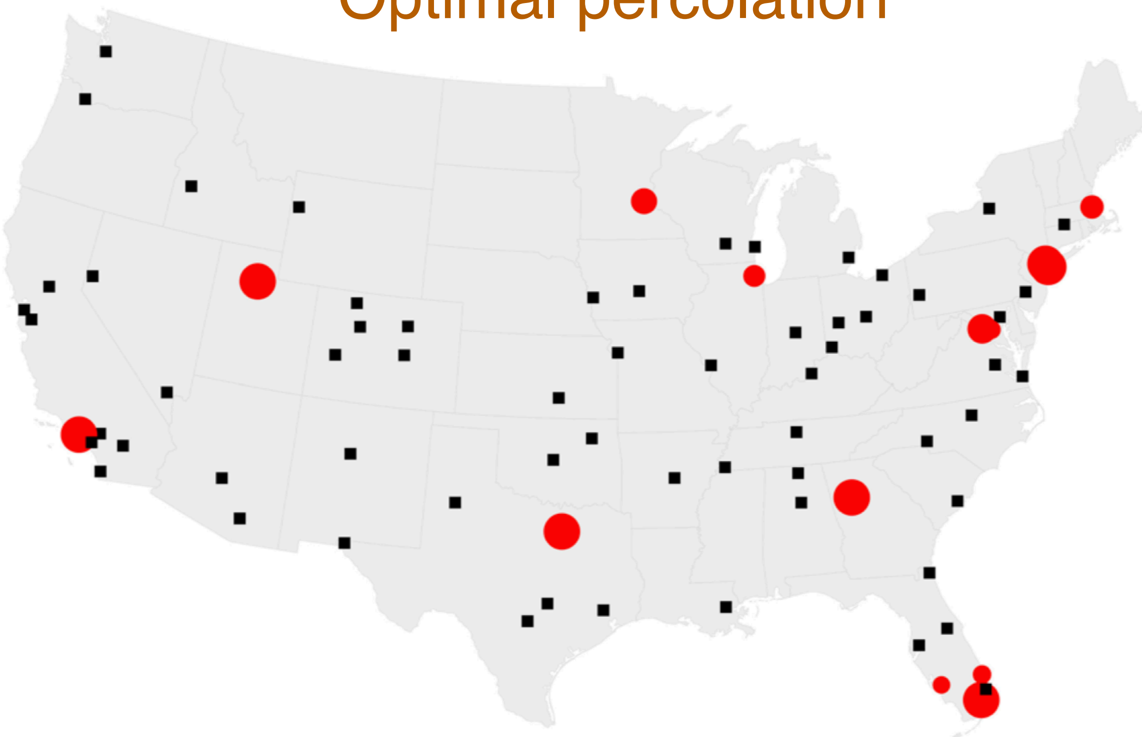
- ✓ Theory of communities in networks: definition and statistical significance
- ✓ Algorithms for community detection
- ✓ Benchmark graphs for testing community detection algorithms

Interdependent networks

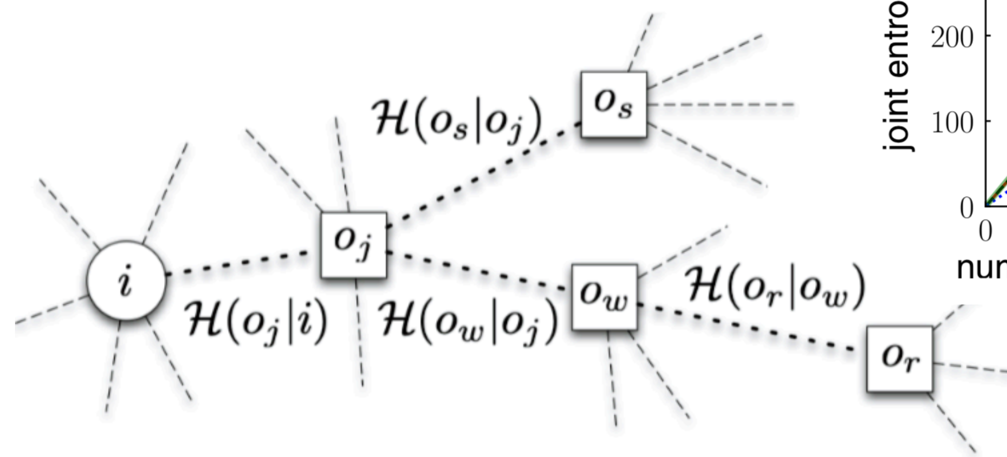


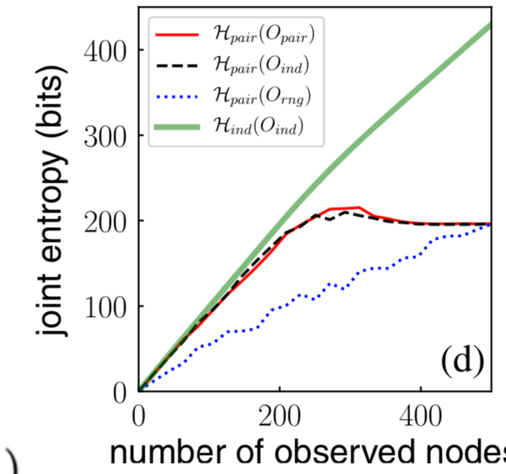
Optimization problems on complex networks

Optimal percolation

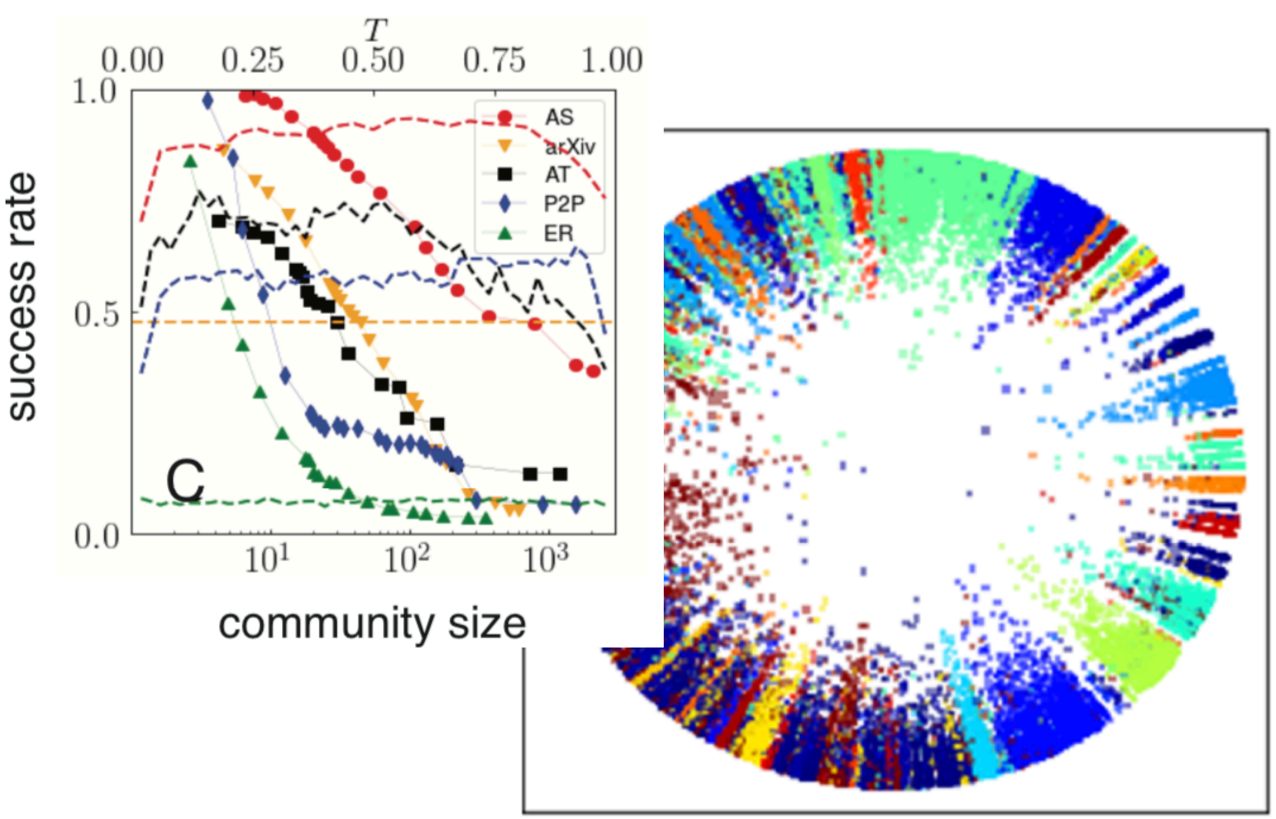


Optimal sampling

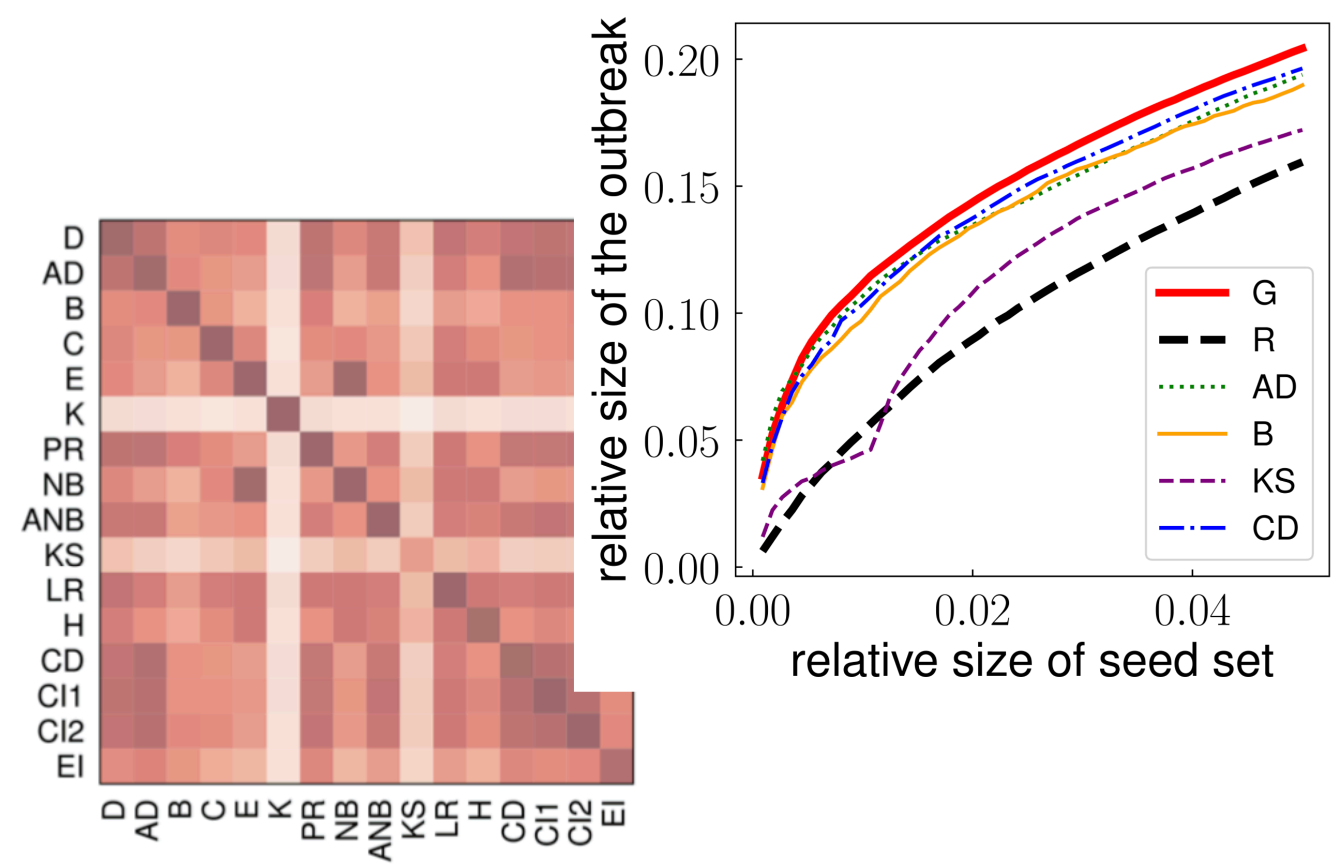




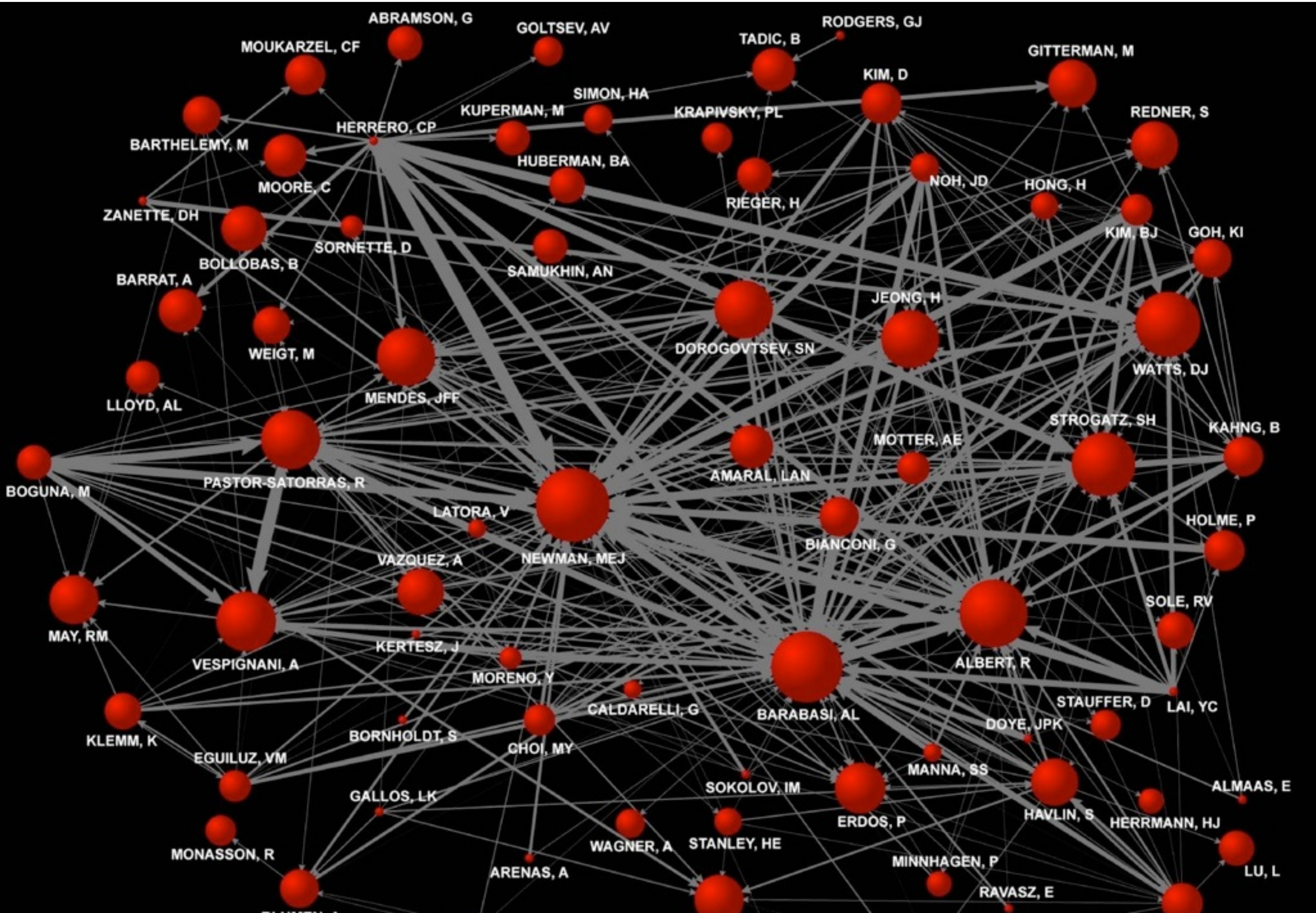
Optimal navigation



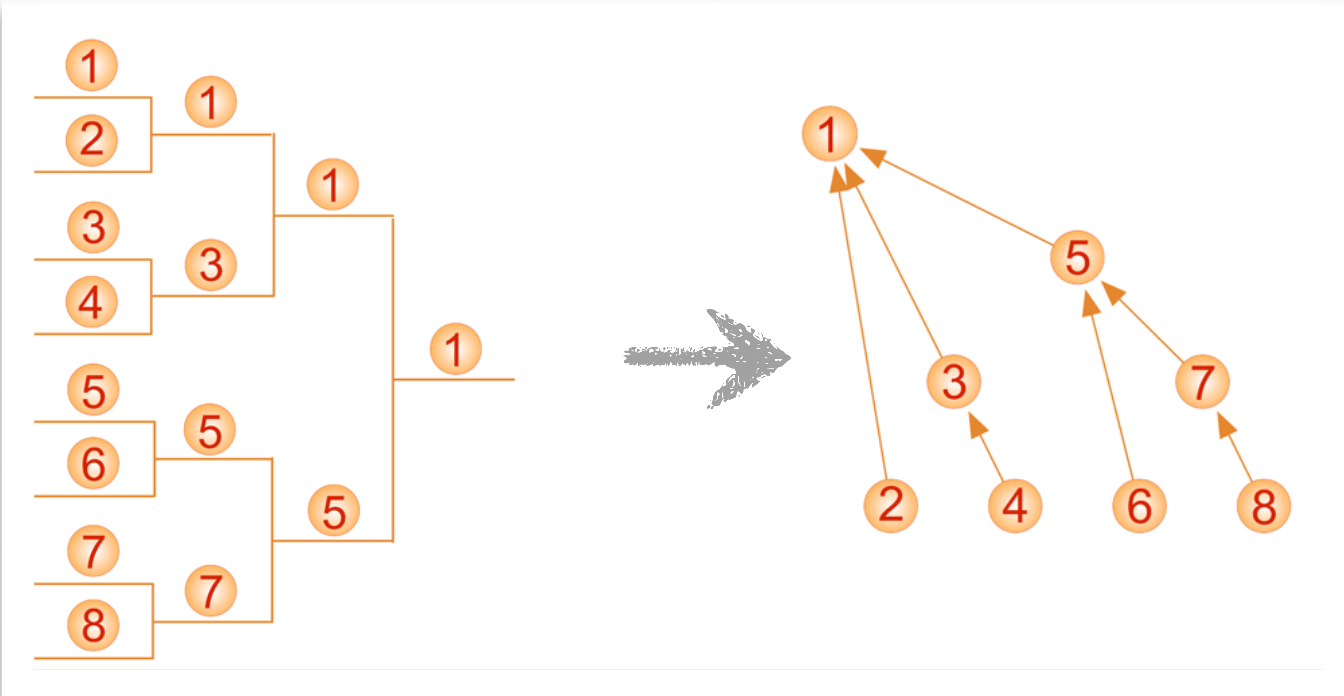
Optimal seeding

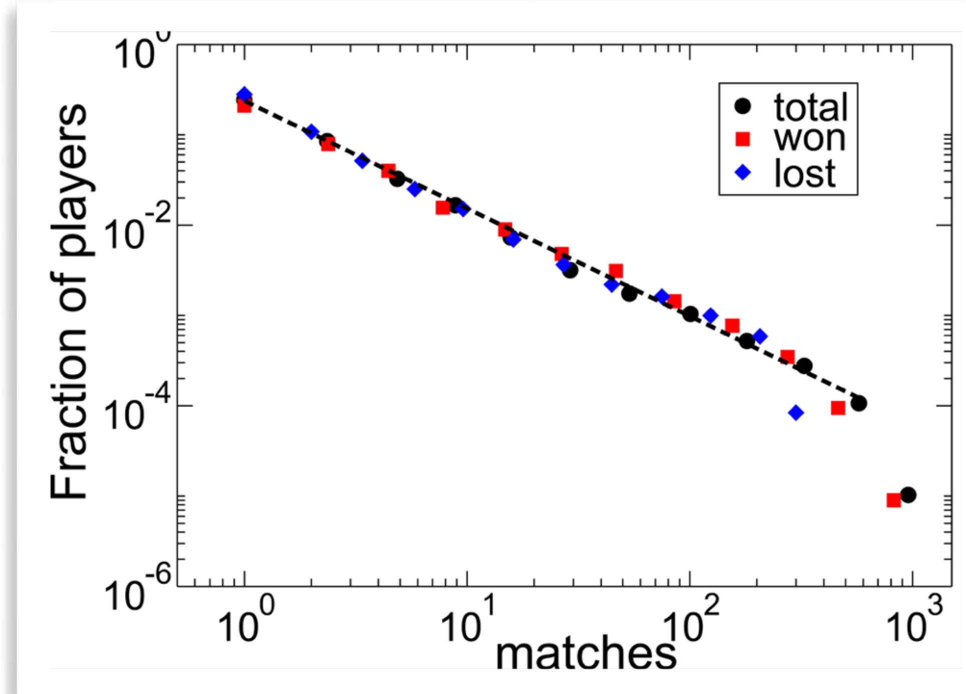


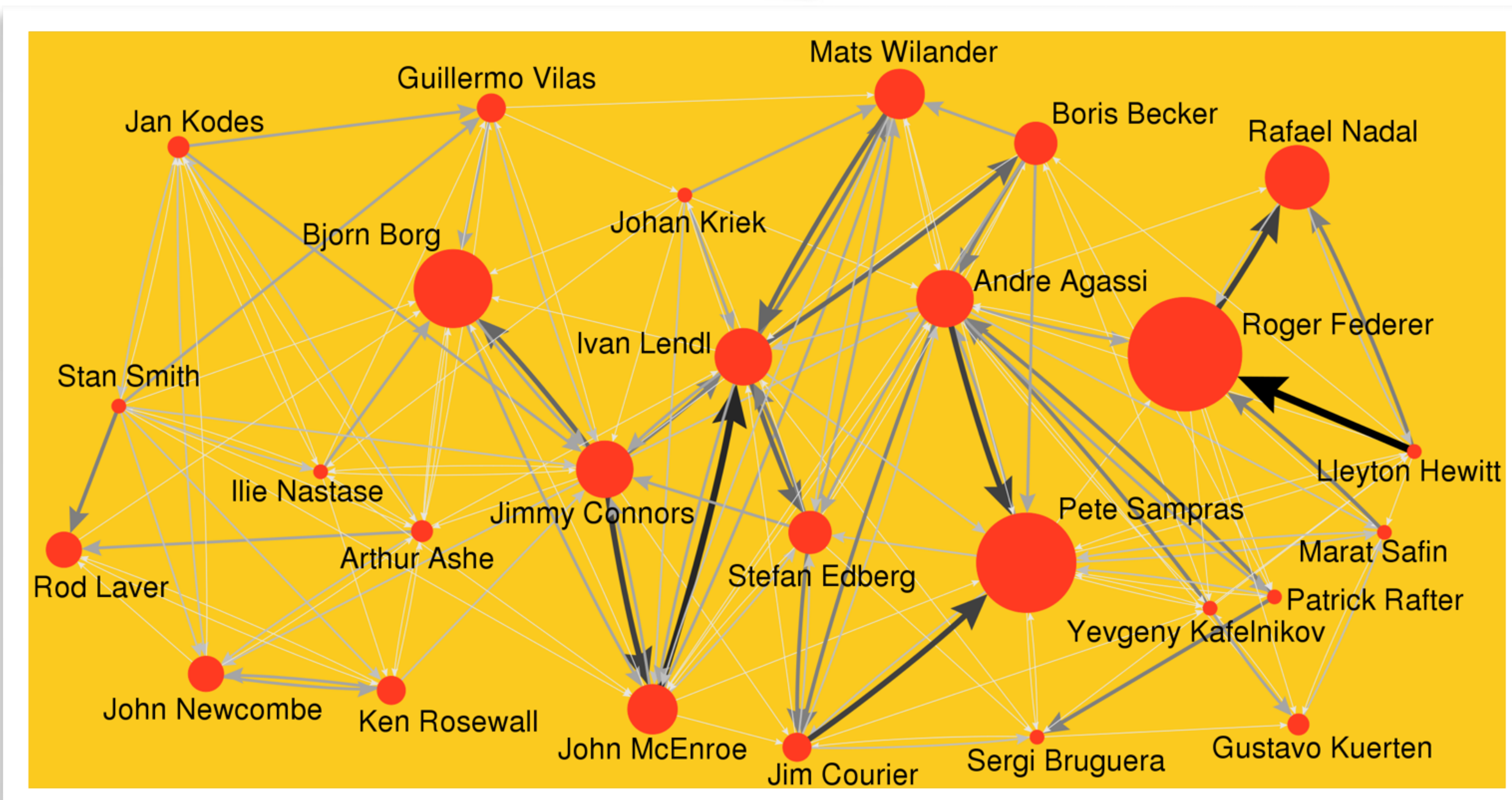
Science of science



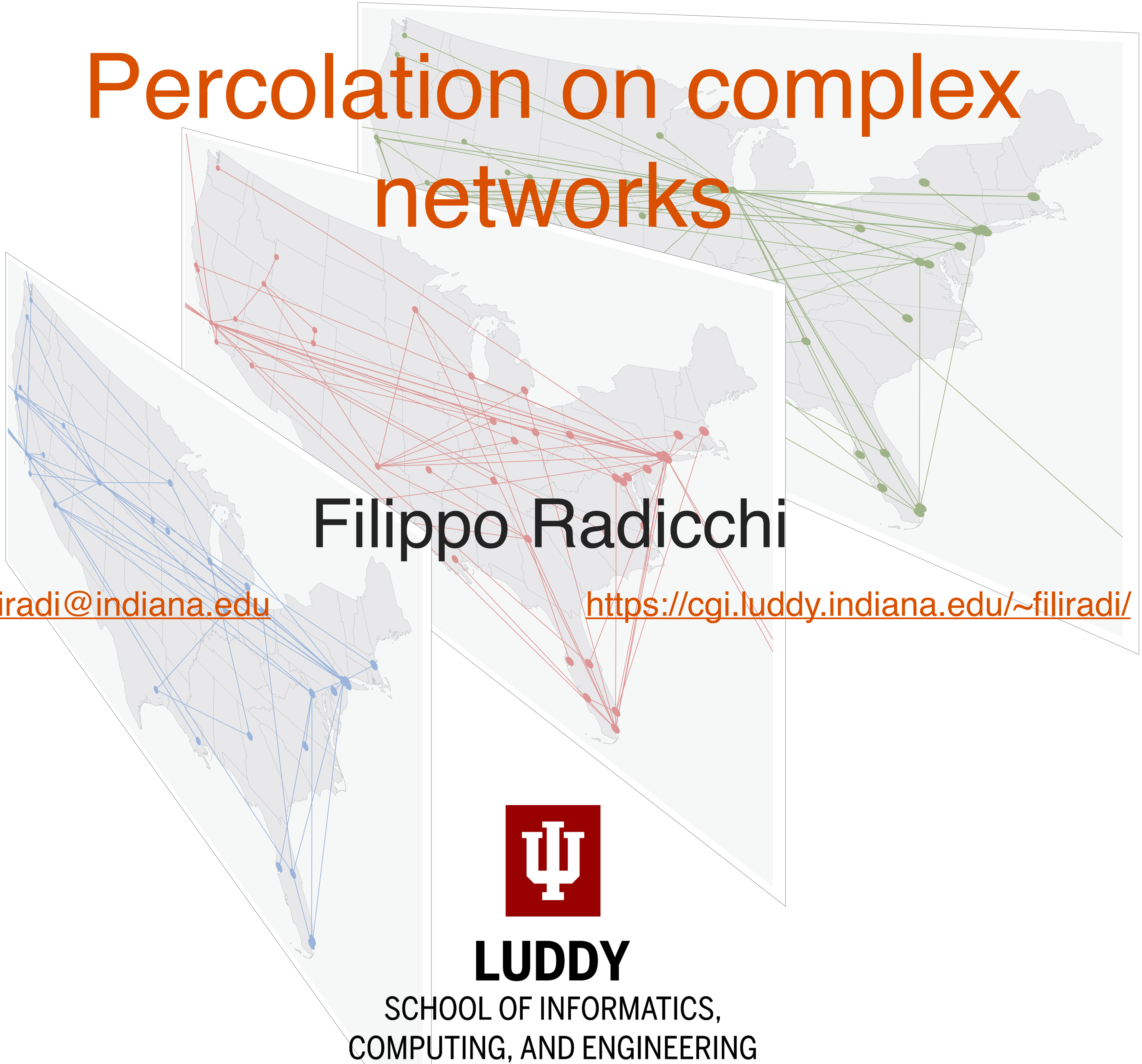
Sports data and performance evaluation







Percolation on complex networks



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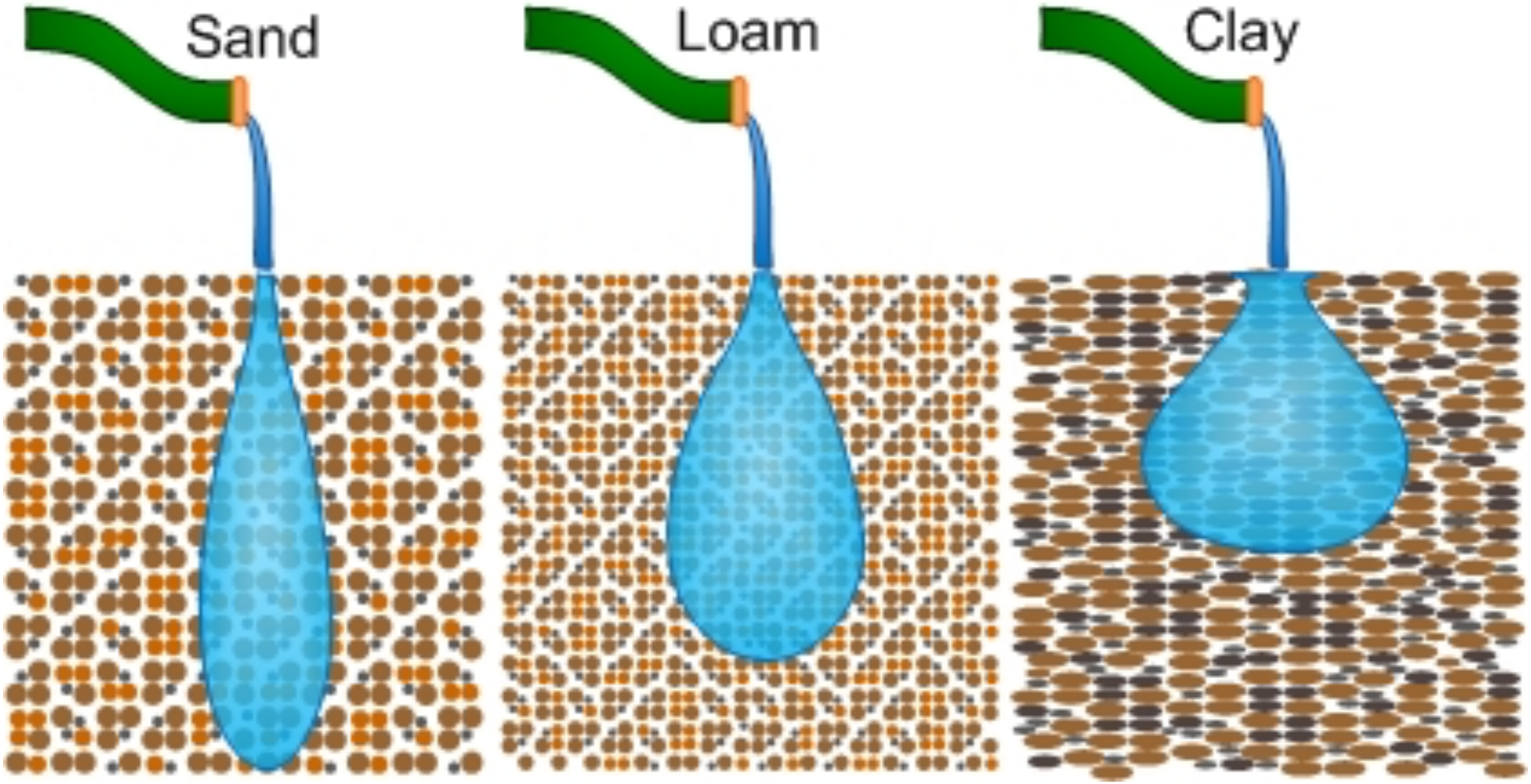
What is percolation?

percolation refers to the movement and/or filtering of fluids through porous materials



Coffee percolators

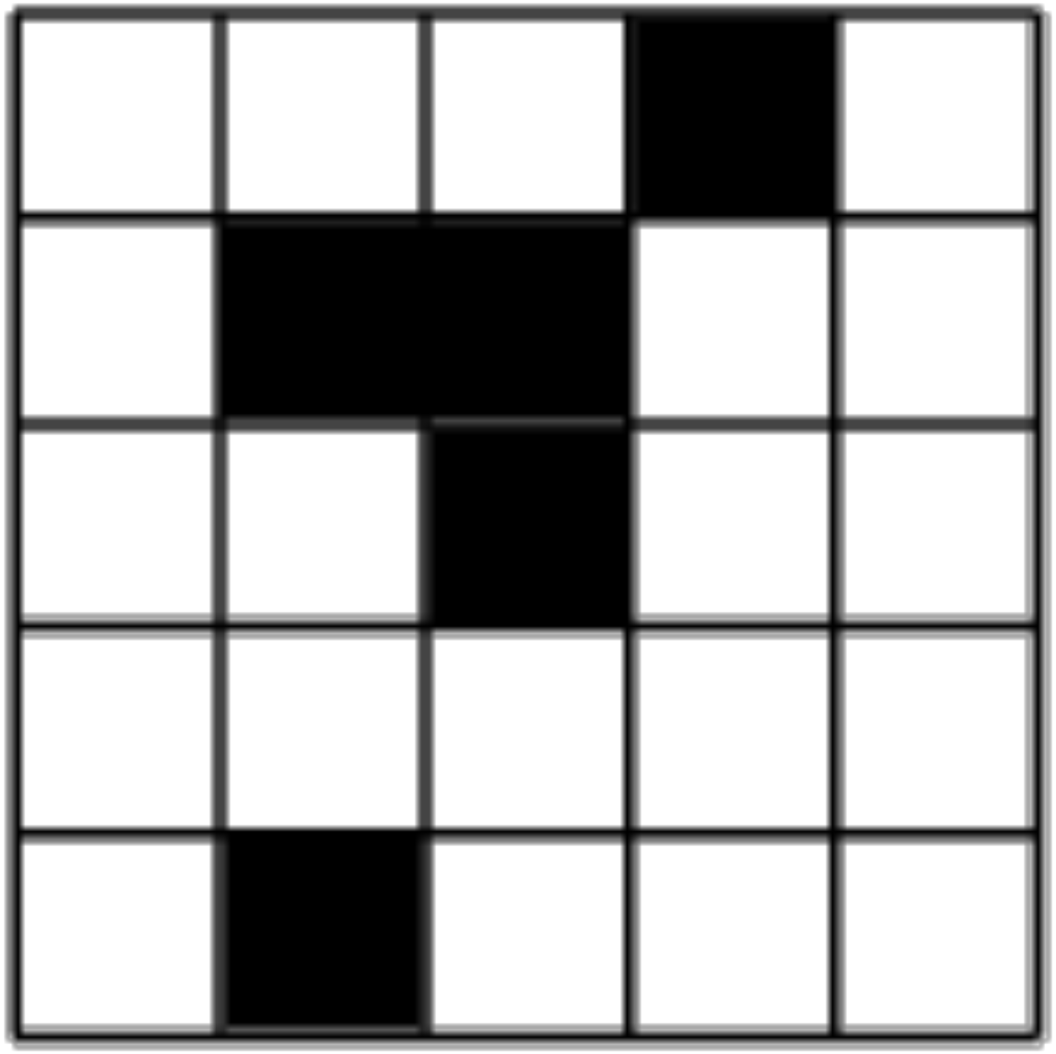
Filter machine

Neapolitan machine

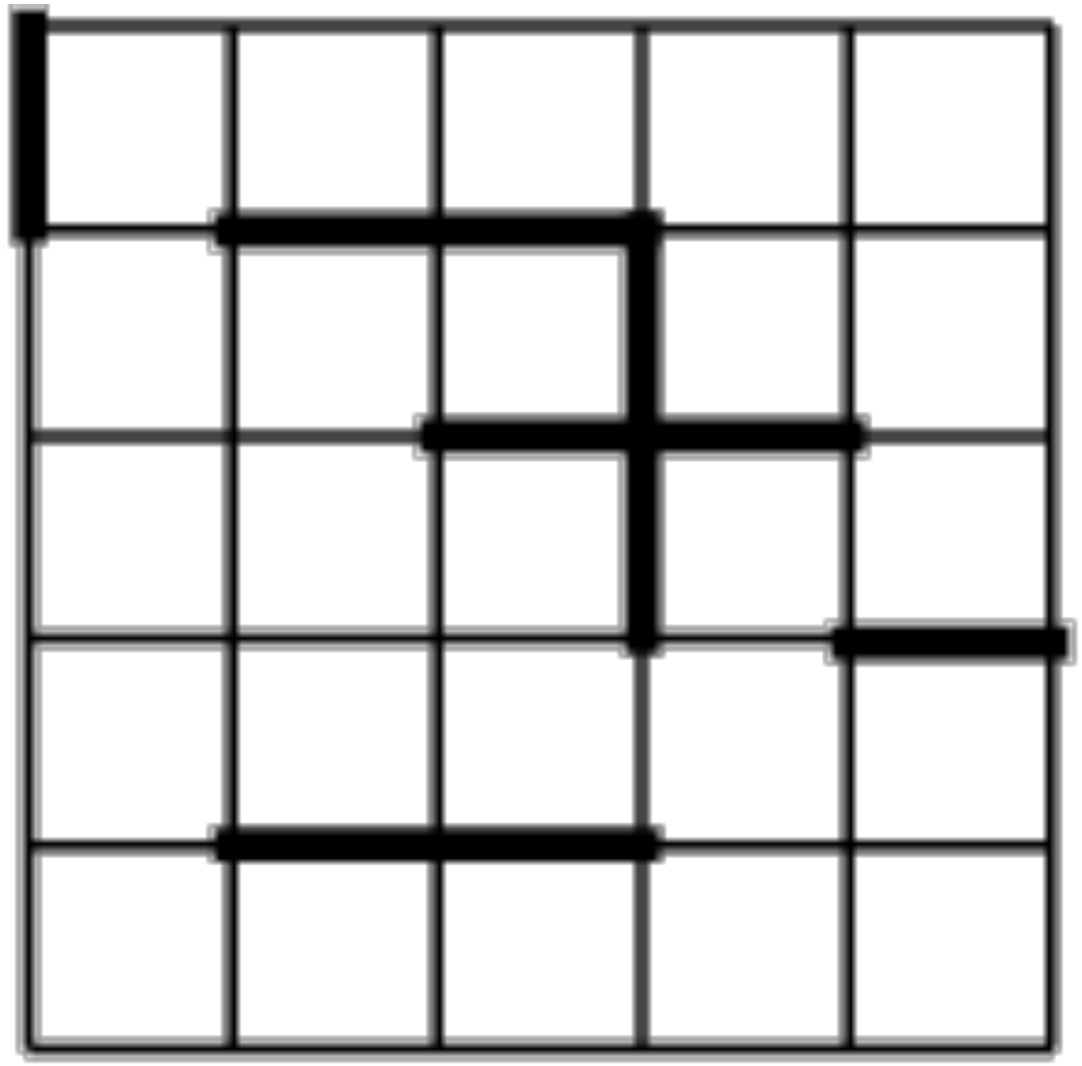
Mocha machine

Espresso machine

Ordinary percolation models



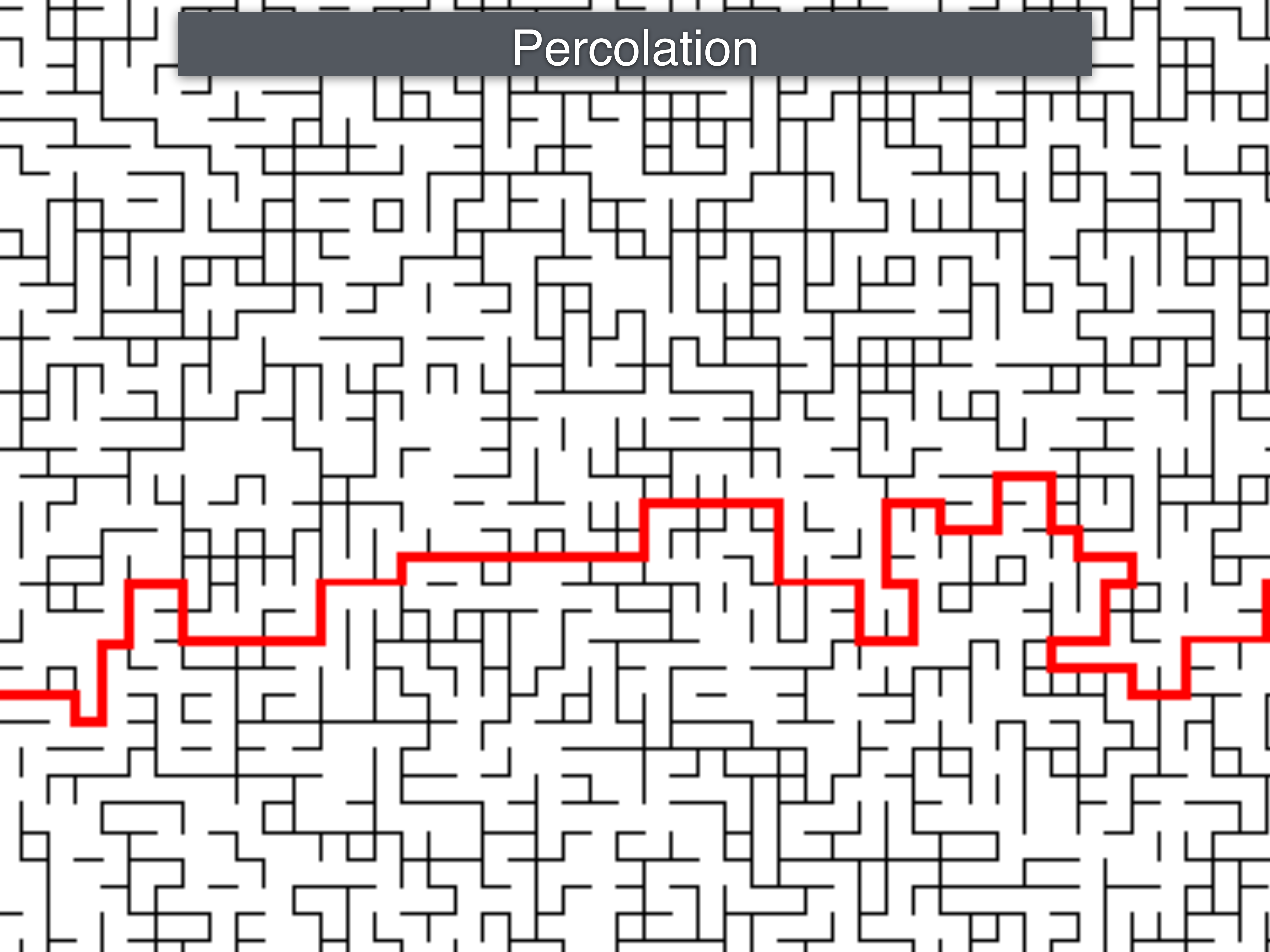
site percolation



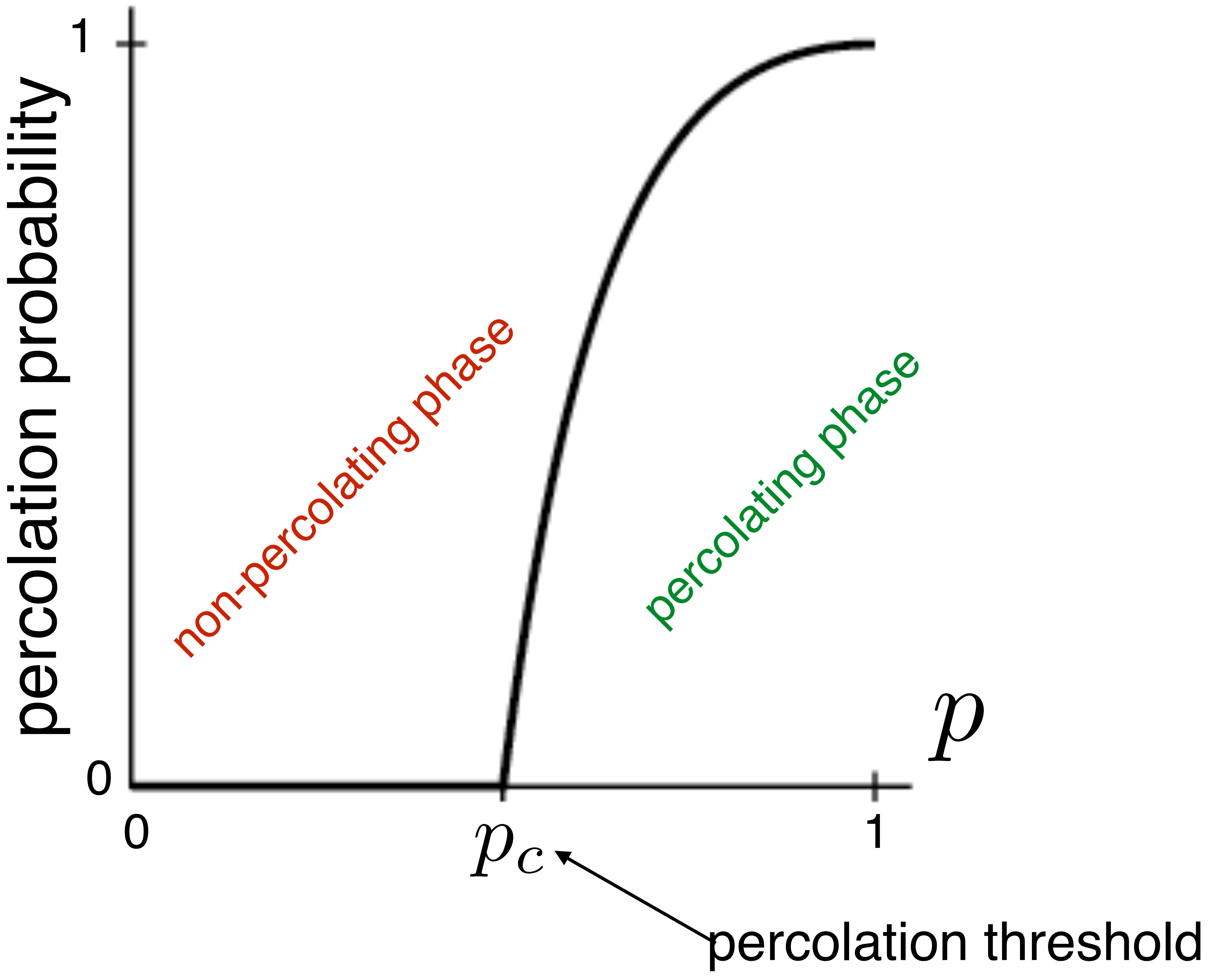
bond percolation

sites or bonds are occupied with probability p

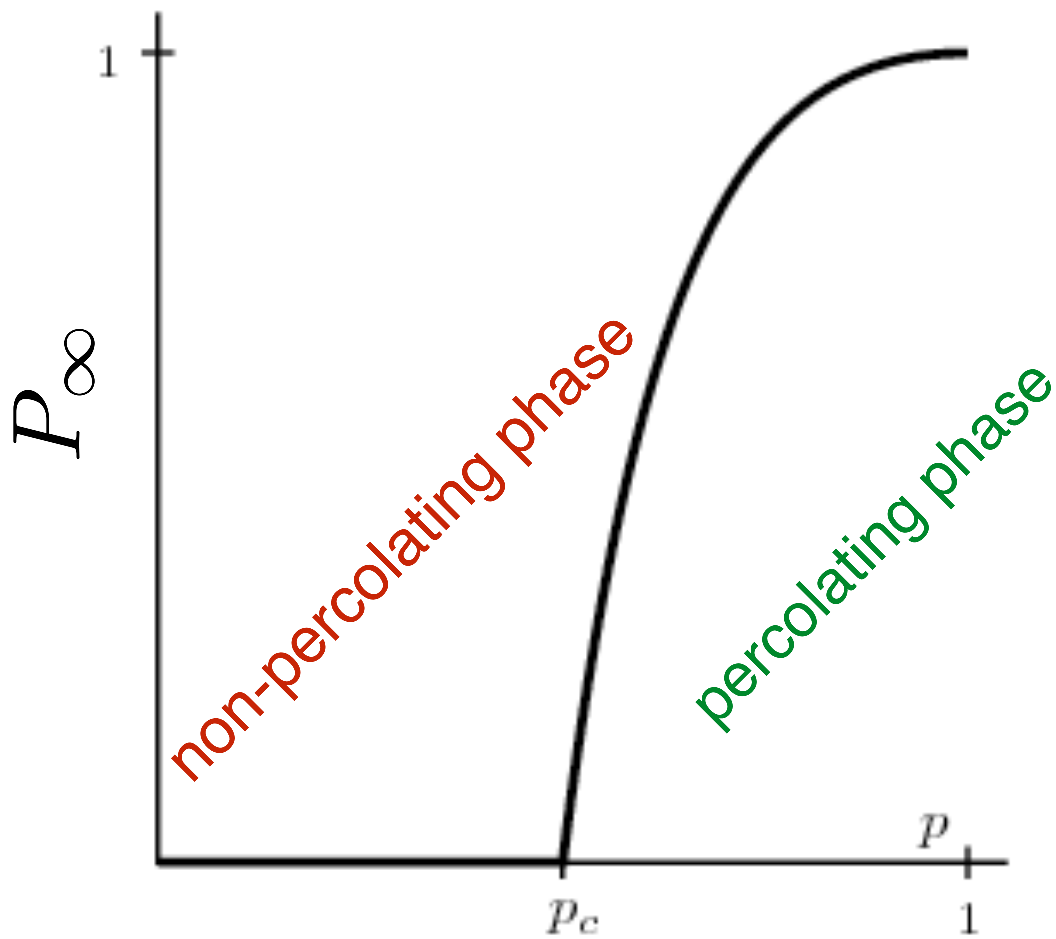
Percolation



Percolation transition



Percolation transition



infinite size

perc. probability $P_\infty \sim (p - p_c)^\beta$

susceptibility $\chi \sim (p - p_c)^{-\gamma}$

correlation length $\xi \sim (p - p_c)^{-\nu}$

$\vdots \quad\quad \vdots$

finite size

$p_c(L)$ max of susceptibility

$p_c(L) - p_c \sim L^{-1/\nu}$

at criticality $P_\infty(L) \sim L^{-\beta/\nu}$

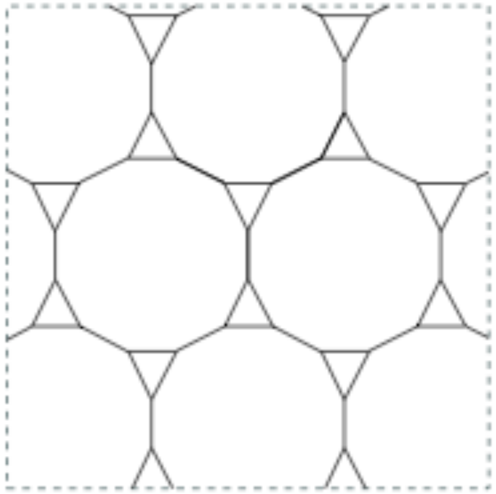
at criticality $\chi(L) \sim L^{\gamma/\nu}$

$\vdots \quad\quad \vdots$

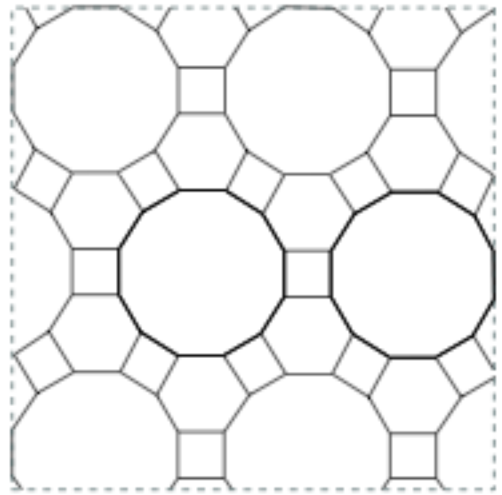
Stauffer, D., & Aharony, A. (1994). *Introduction to percolation theory*. CRC press.

Percolation thresholds

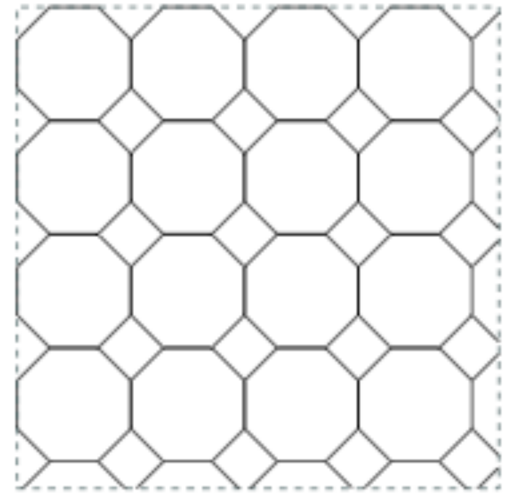
values of bond and site percolation thresholds depend on the geometry of the system



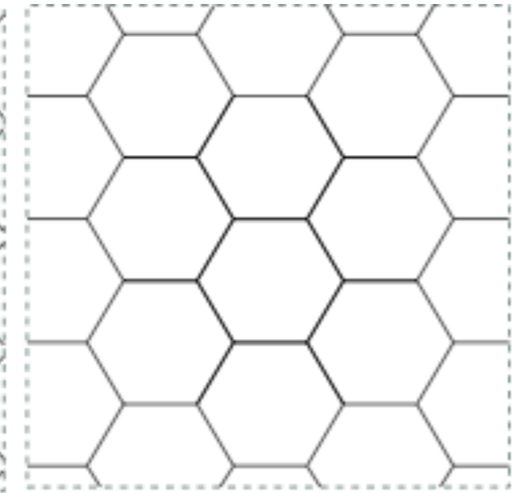
$(3, 12^2)$



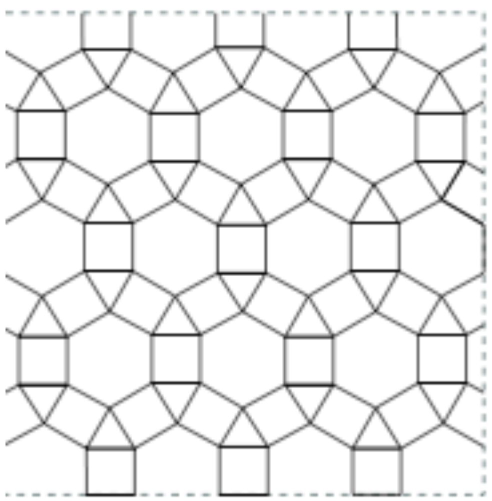
$(4, 6, 12)$



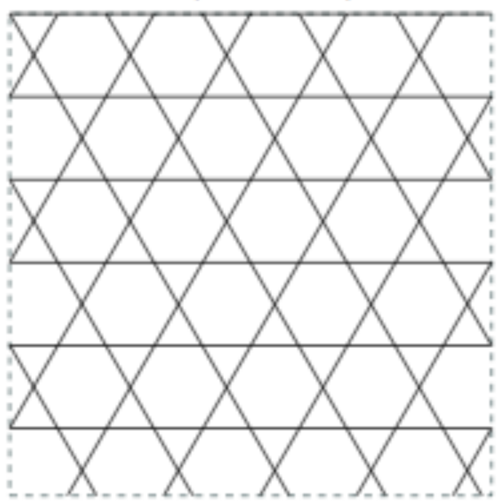
$(4, 8^2)$



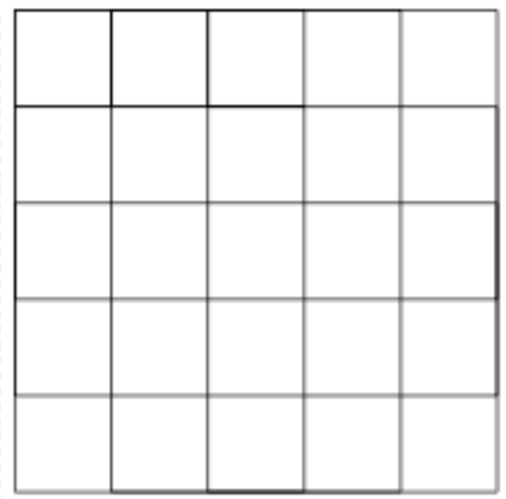
(6^3)



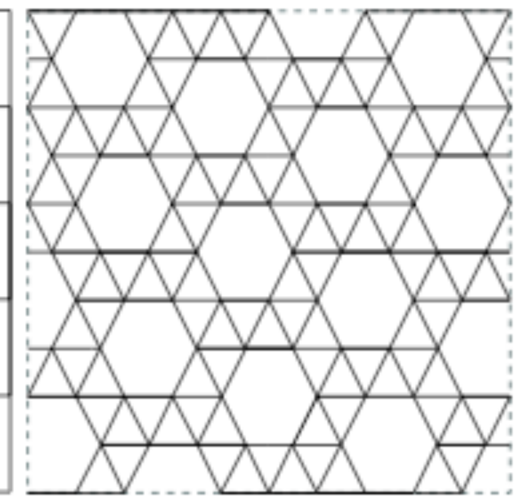
$(3, 4, 6, 4)$



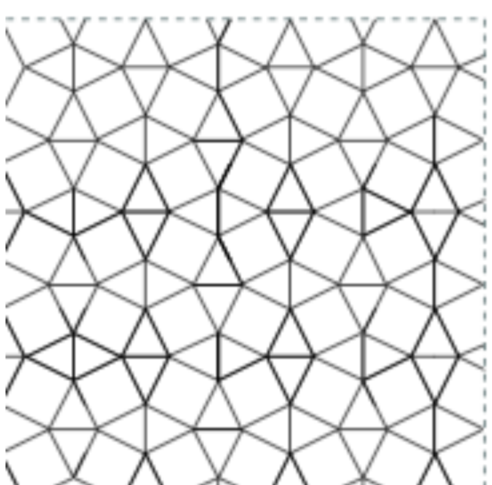
$(3, 6, 3, 6)$



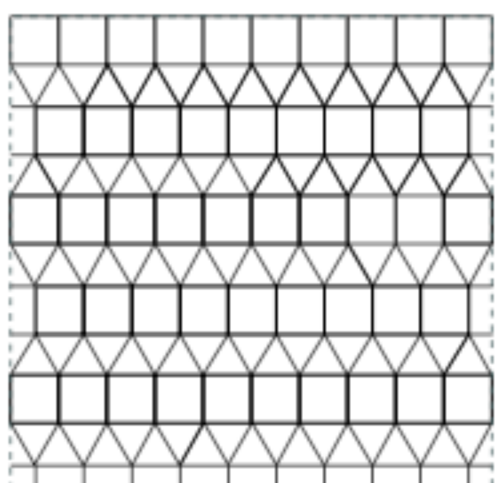
(4^4)



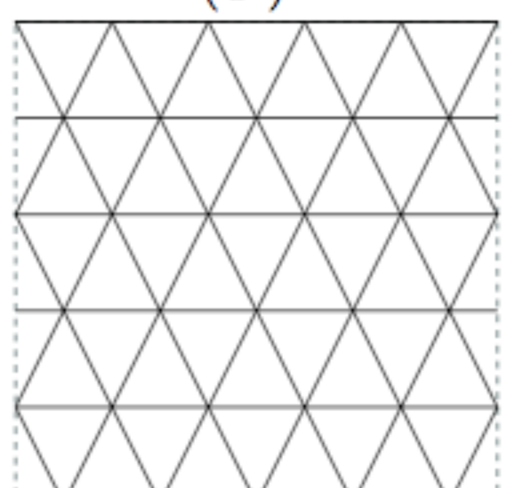
$(3^4, 6)$



$(3^2, 4, 3, 4)$



$(3^3, 4^2)$



(3^6)

Percolation universality classes

bond and site percolation have the same critical exponents

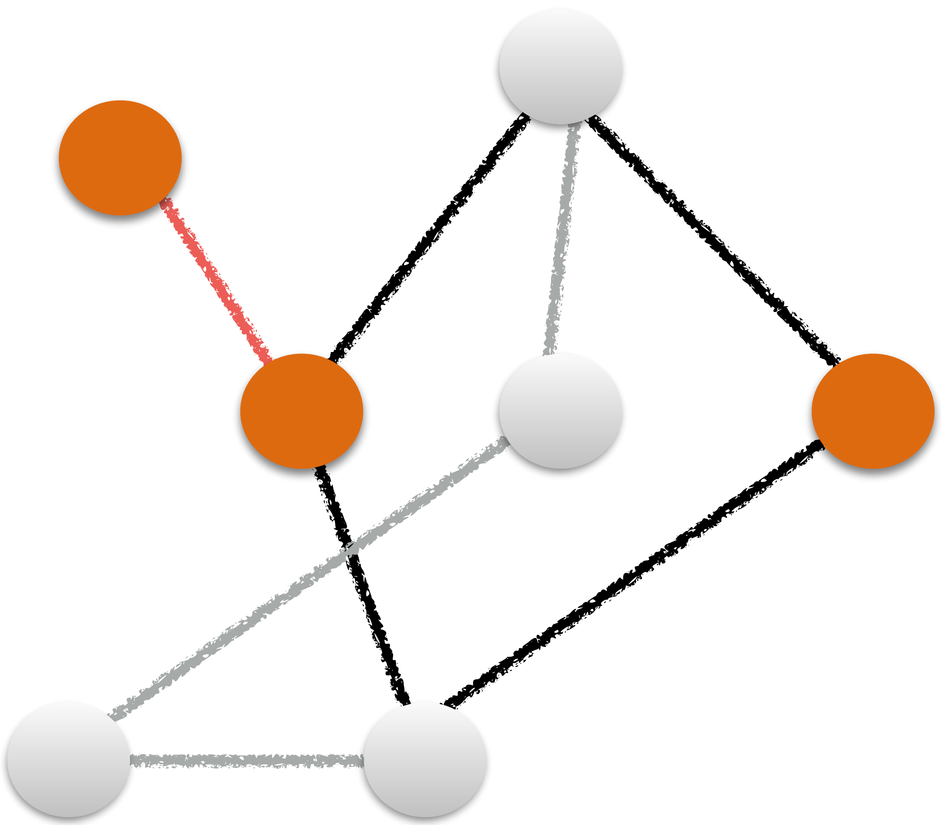
https://en.wikipedia.org/wiki/Percolation_critical_exponents

Exponents for standard percolation [edit]

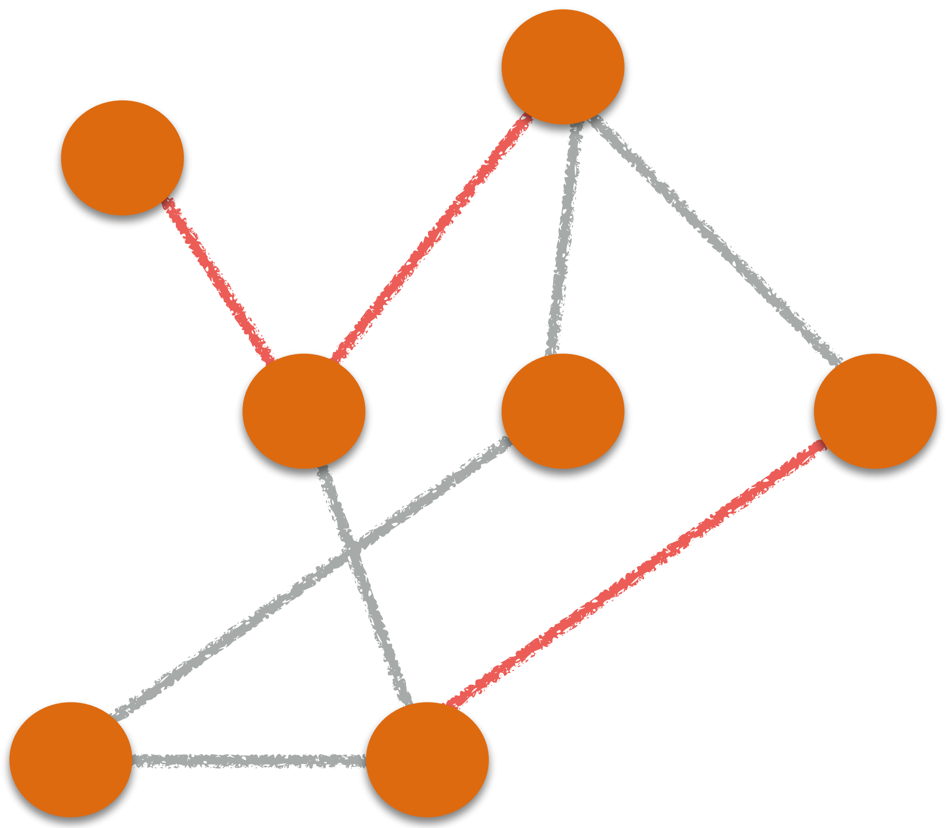
d	1[13]	2	3	4	5	$6-\varepsilon$[14][15][16][note 1]	6 +
α	1	−2/3	-0.625(3) -0.64(4)[19]	-0.756(40) -0.75(2)[19]	-0.870(1)[19]	$-1+\frac{\varepsilon}{7}-\frac{443}{2^2 3^2 7^3}\varepsilon^2$	-1
β	0	0.14(3) [20] 5/36	0.39(2)[21] 0.4181(8) 0.41(1) [22] 0.405(25),[23] 0.4273[18] 0.4053(5)[24] 0.429(4) [19]	0.52(3)[21] 0.639(20)[23] 0.657(9) 0.6590[18] 0.658(1) [19]	0.66(5)[21] 0.835(5)[23] 0.830(10) 0.8457[18] 0.8454(2)[19]	$1-\frac{\varepsilon}{7}-\frac{61}{2^2 3^2 7^3}\varepsilon^2$	1
γ	1	43/18	1.6[22] 1.80(5) [21] 1.66(7) [25] 1.793(3) 1.805(20) [26] 1.8357[18] 1.819(3)[24]	1.6(1) [21] 1.48(8)[25] 1.422(16) 1.4500[18] 1.435(15)[26] 1.430(6)[19]	1.3(1)[21] 1.18(7)[25] 1.185(5) [26] 1.1817[18] 1.1792(7) [19]	$1+\frac{\varepsilon}{7}+\frac{565}{2^2 3^2 7^3}\varepsilon^2$	1

Ordinary percolation models in networks

site percolation



bond percolation

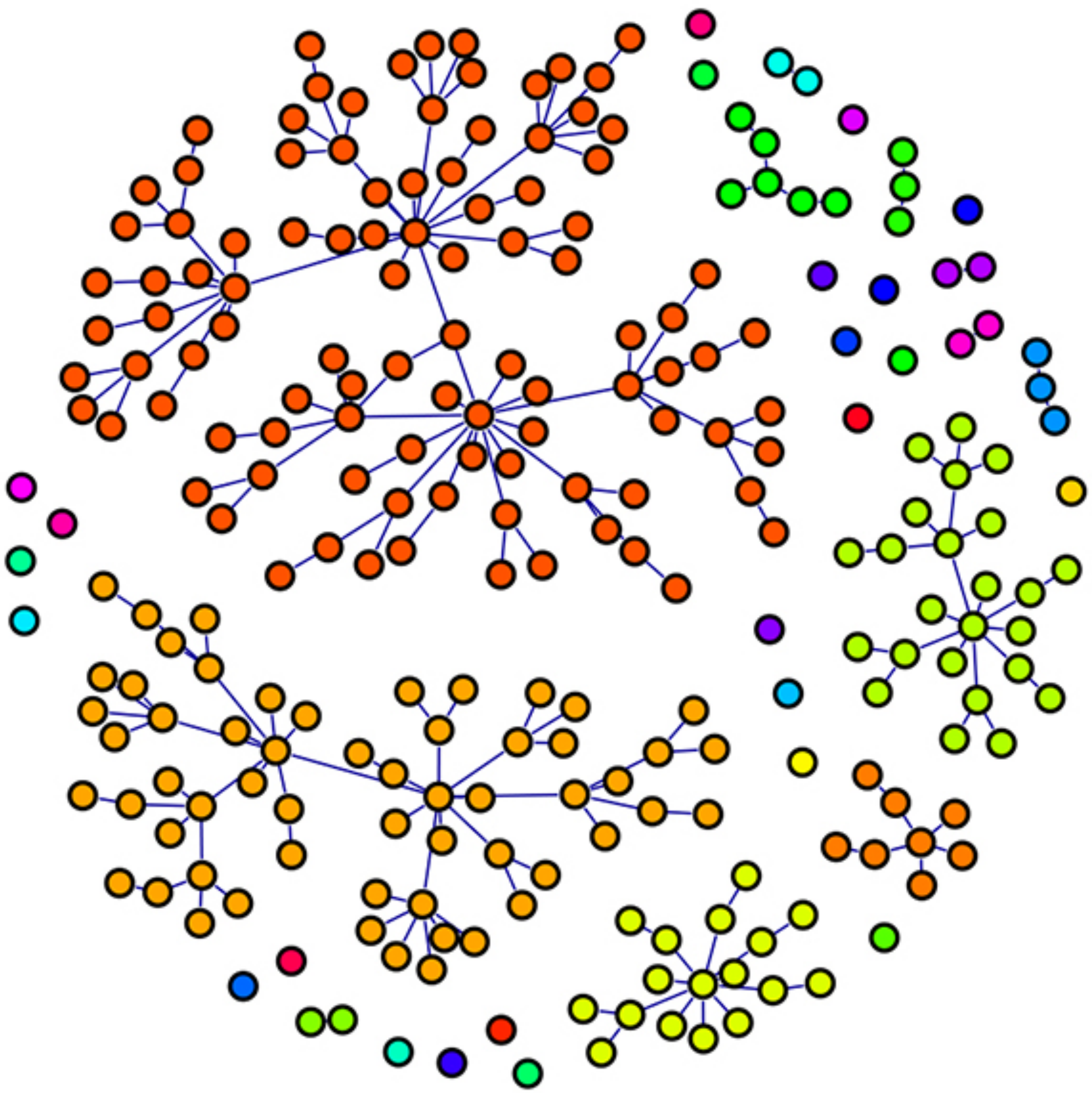


nodes or edges are occupied with probability p

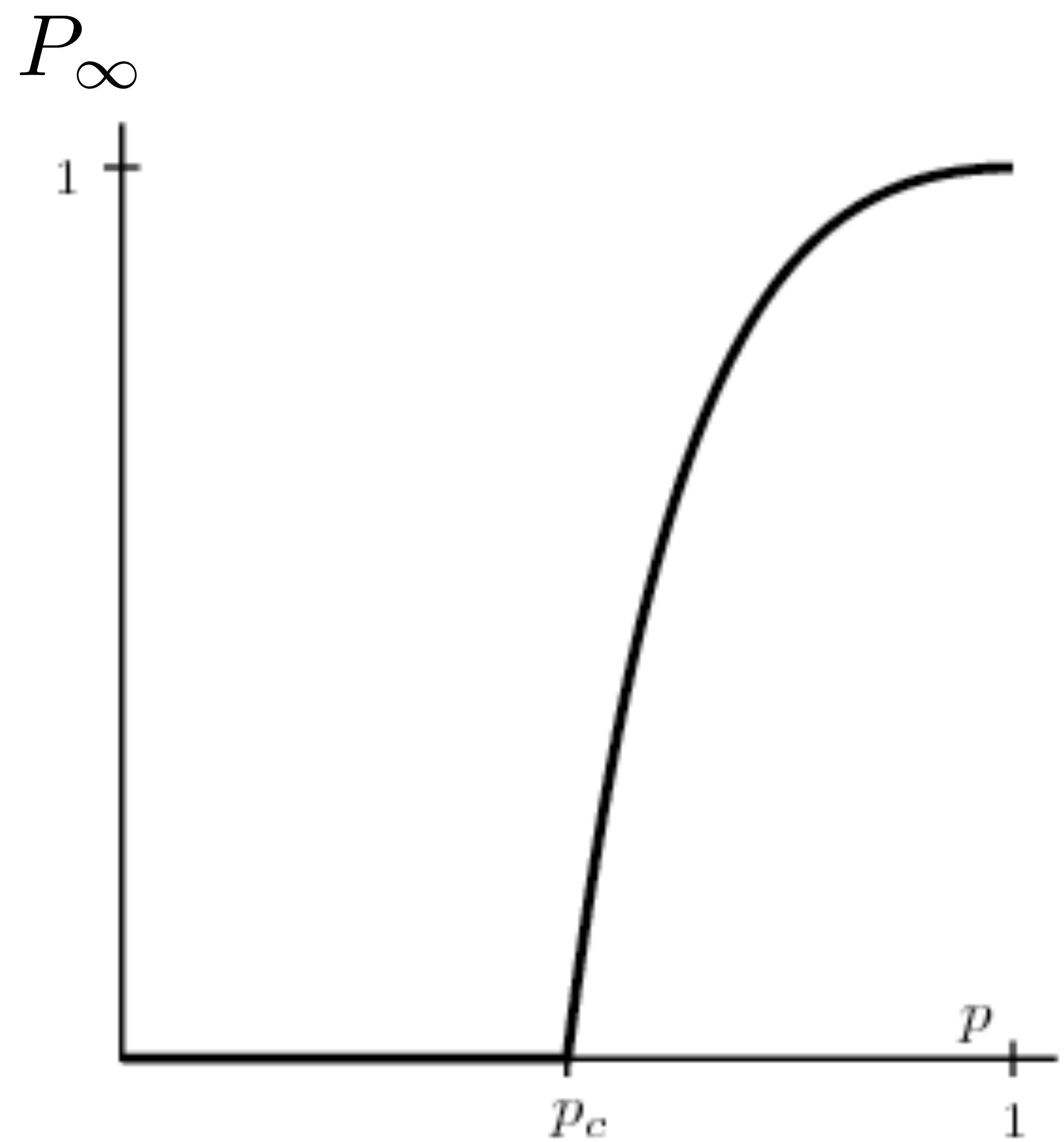
Percolation transition in networks

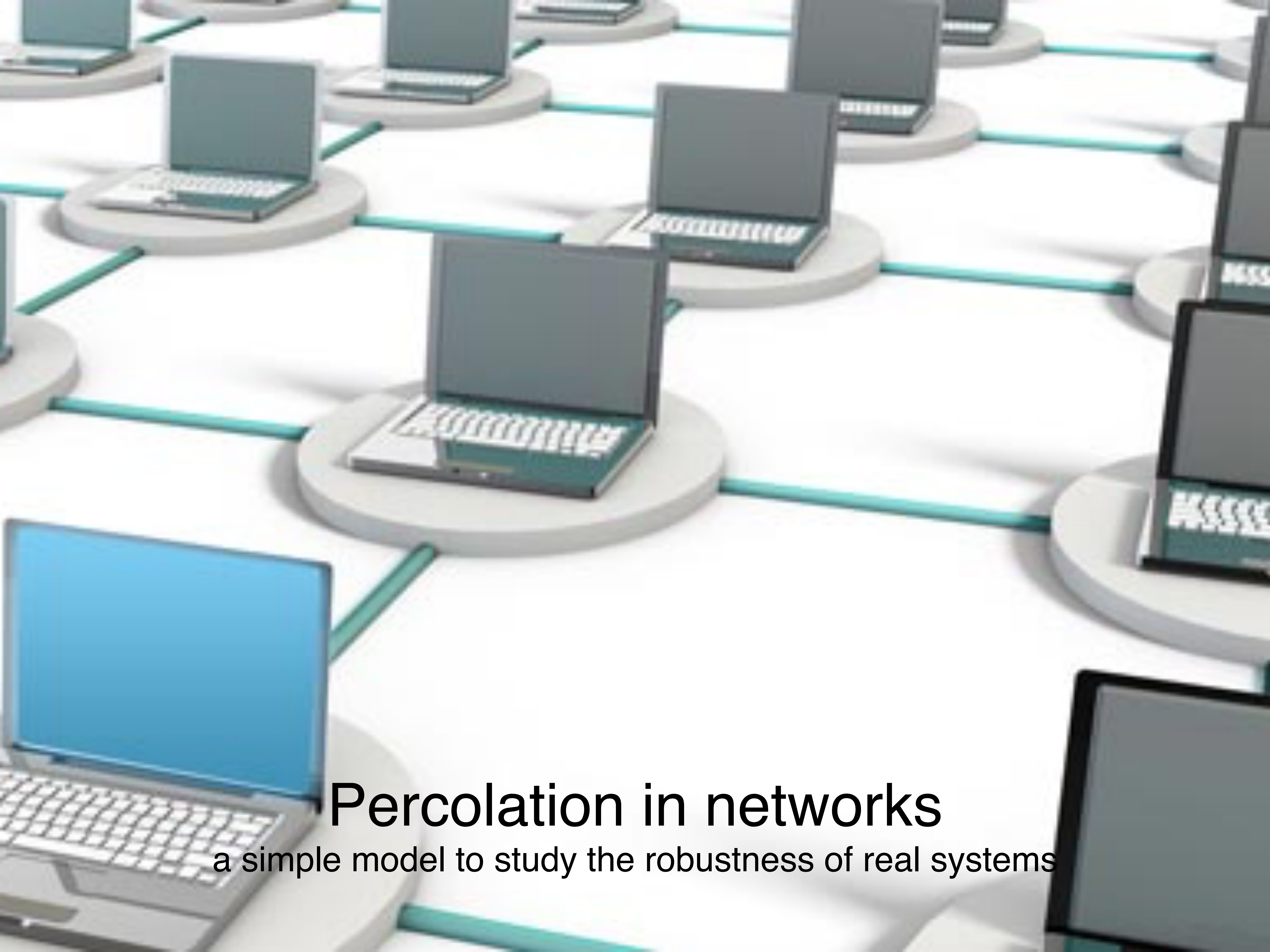
order parameter = percolation strength

= relative size of the largest connected component in the graph



Nature Physics cover, July 2015



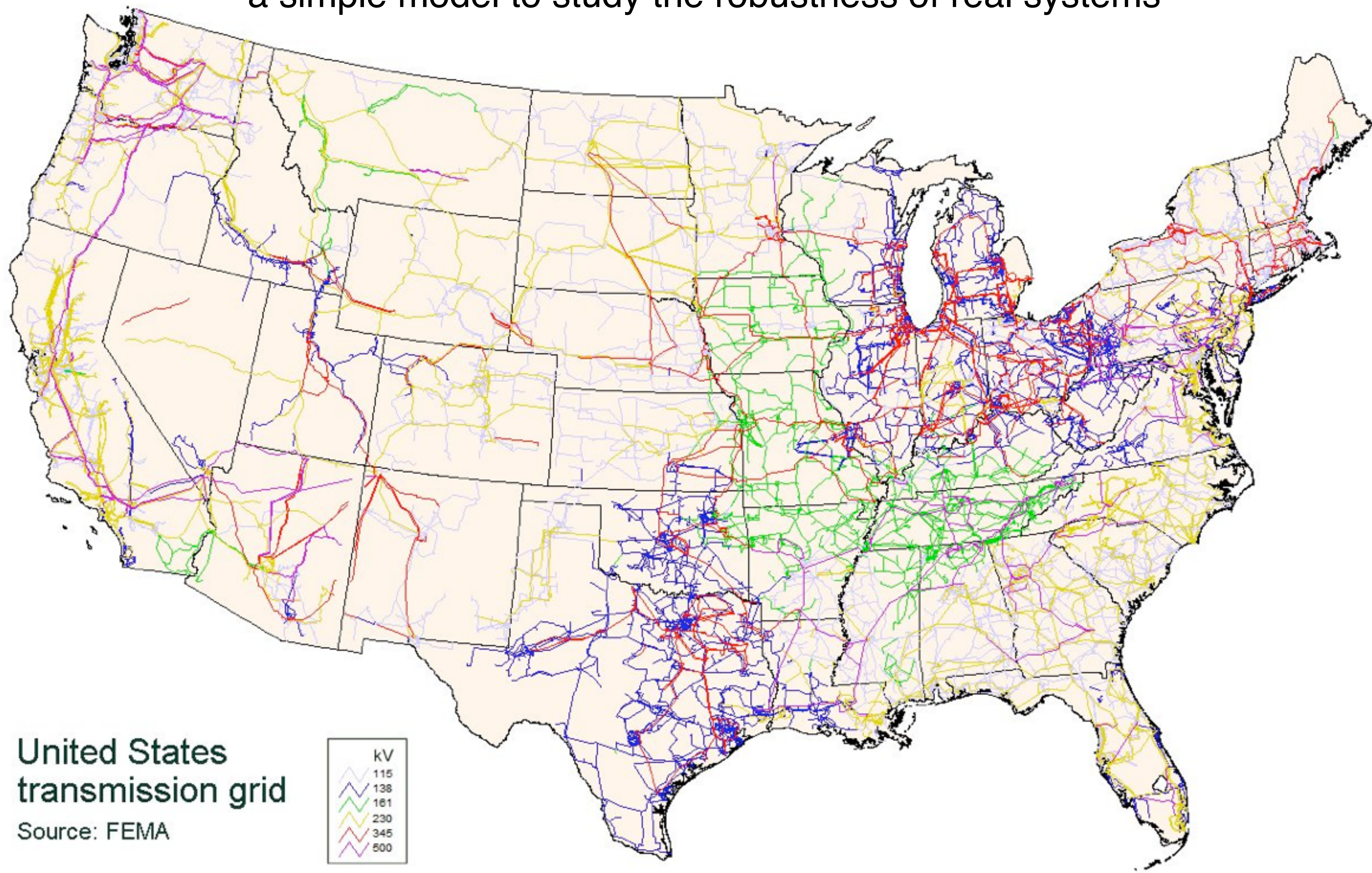


Percolation in networks

a simple model to study the robustness of real systems

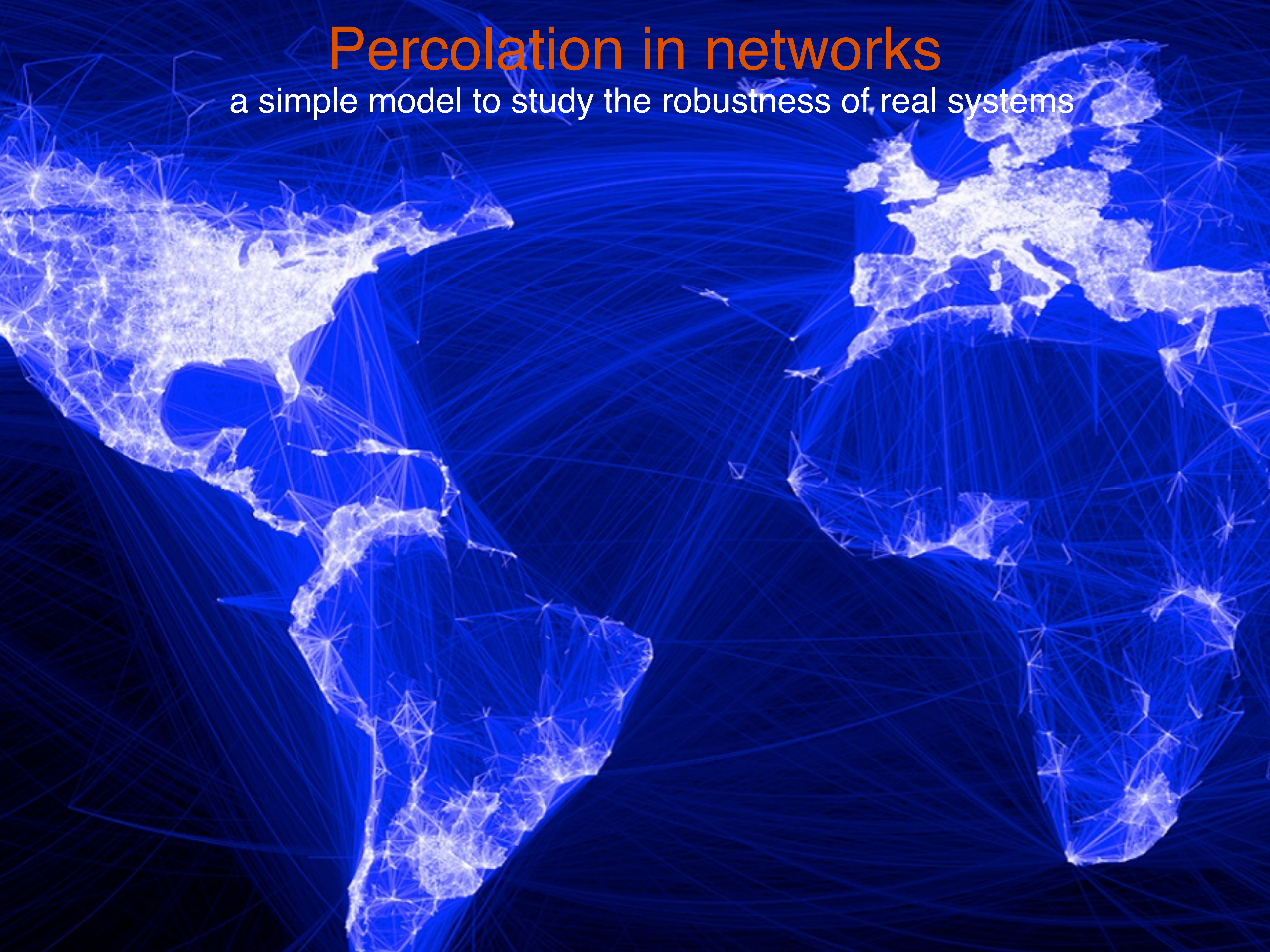
Percolation in networks

a simple model to study the robustness of real systems



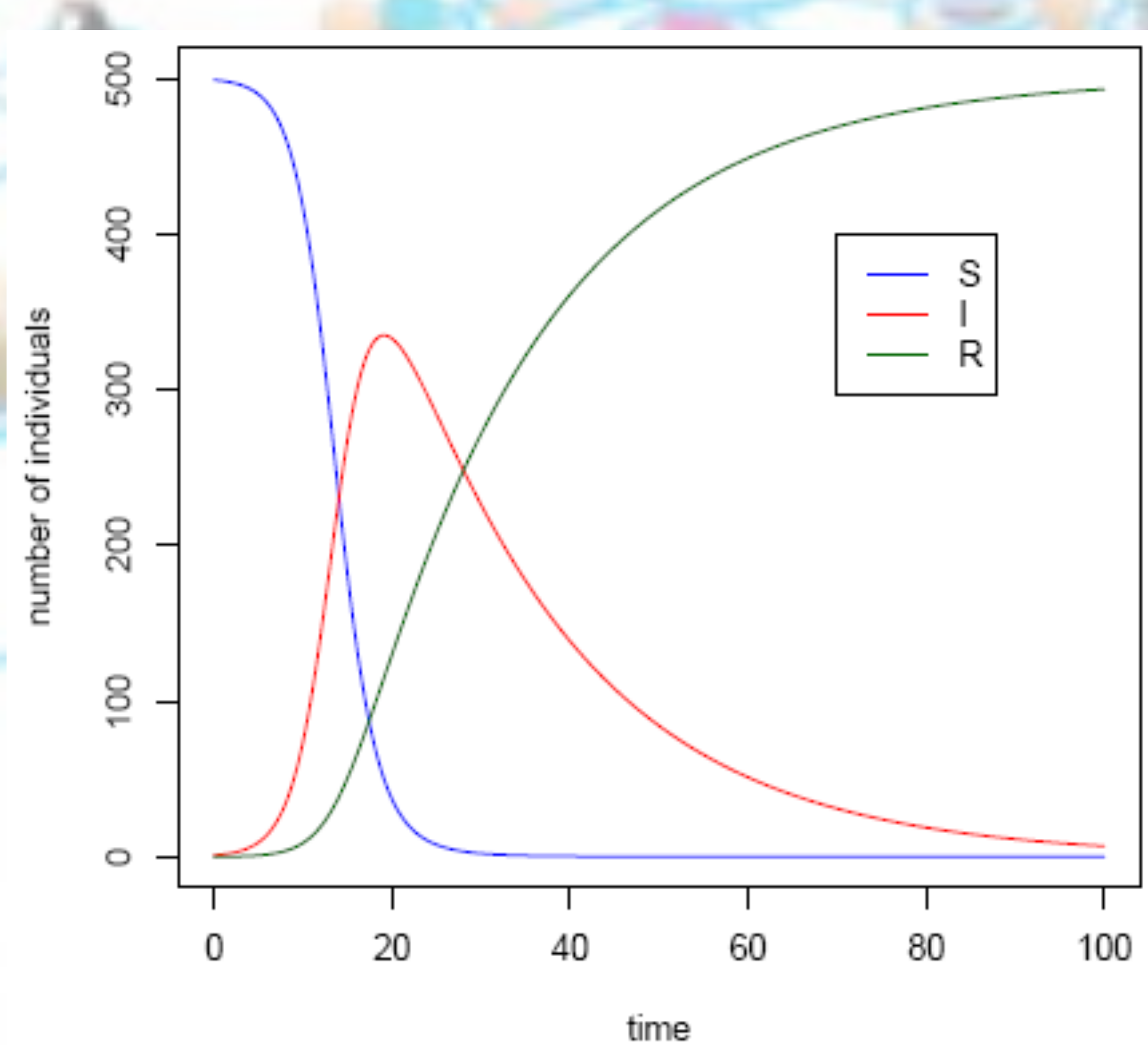
Percolation in networks

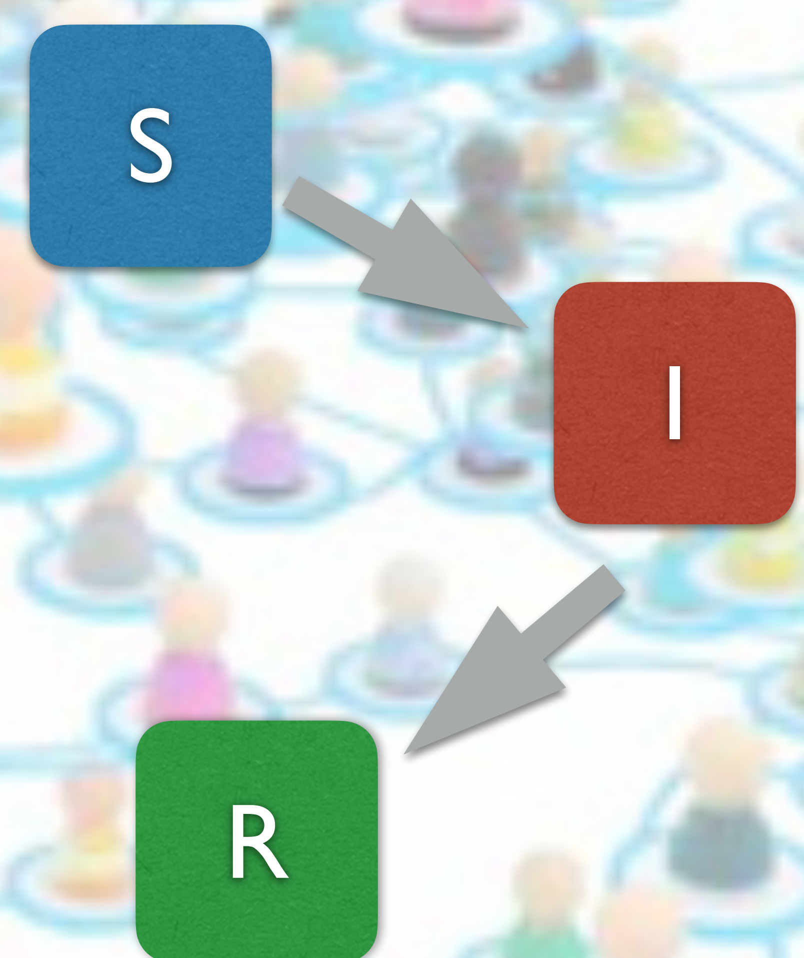
a simple model to study the robustness of real systems



Percolation in networks

strict analogy with simple epidemiological models





Grassberger, P. On the critical behavior of the general epidemic process and dynamical percolation. Math. Biosci. 63, 157–172 (1983).

https://github.com/filrad/Percolation-on-complex-networks

Naive implementation of the ordinary bond-percolation model

Inputs of the model are a graph and the occupation probability p.

1. For each edge in the graph, generate a random number r from the uniform distribution U(0,1). If $r > p$, the edge is removed from the graph. This procedure generates an instance of the percolation model.

2. Take measurements (i.e., cluster sizes) on the instance of the percolation model obtained at point 1.

Repeat points 1 and 2 for a desired number of times to generate enough statistics; vary the value of p to generate percolation diagrams.

jupyter percolation_on_complex_networks Last Checkpoint: 10 minutes ago (autosaved)

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Percolation on complex networks

This notebook serves at providing a brief introduction to the numerical study of percolation processes in complex networks. Given the large volume of work on this topic, the document illustrates only a small number of percolatio models.

Outline

- Naive implementations of the ordinary bond and site percolation models
- Newman-Ziff algorithm for the ordinary bond and site percolation models
- Message-passing equations for ordinary percolation models on sparse networks
- Explosive percolation
- Optimal percolation
- Ordinary site percolation on multiplex networks
- Message-passing equations for ordinary site percolation on sparse multiplex networks

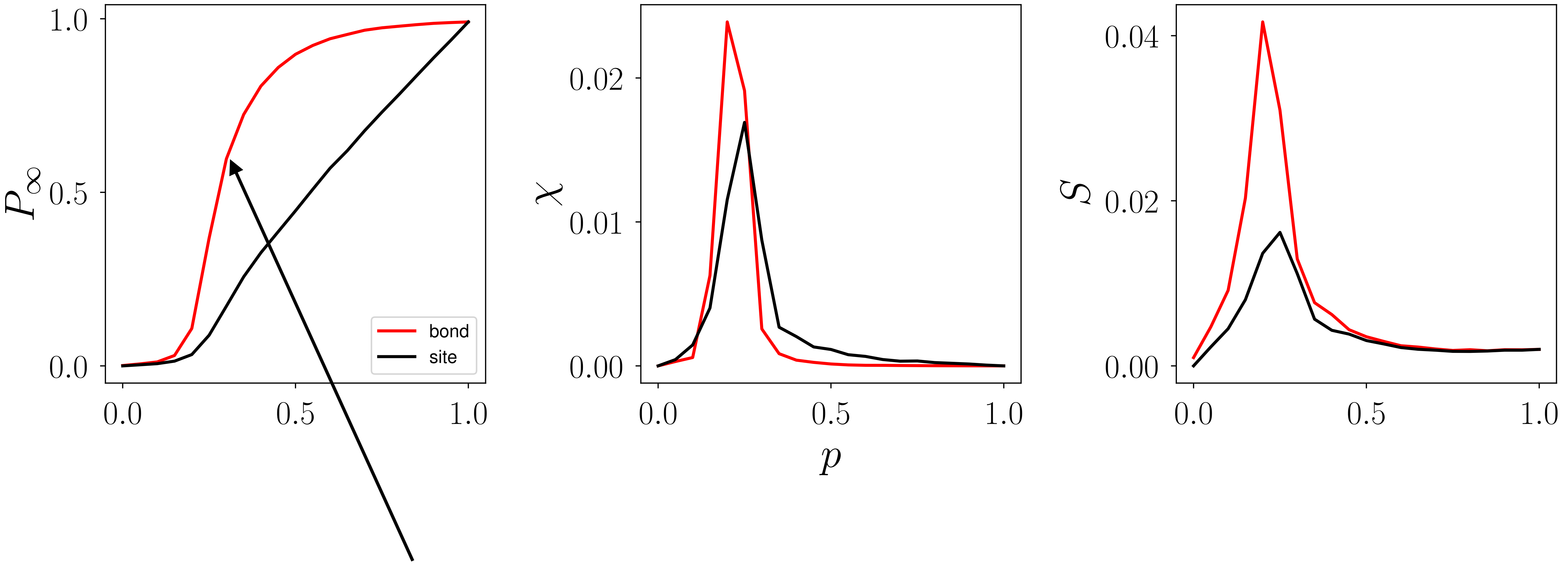
Libraries used in this notebook

In [1]:

```
import networkx as nx
import random
import numpy as np
```

Naive implementation of the ordinary bond-percolation model

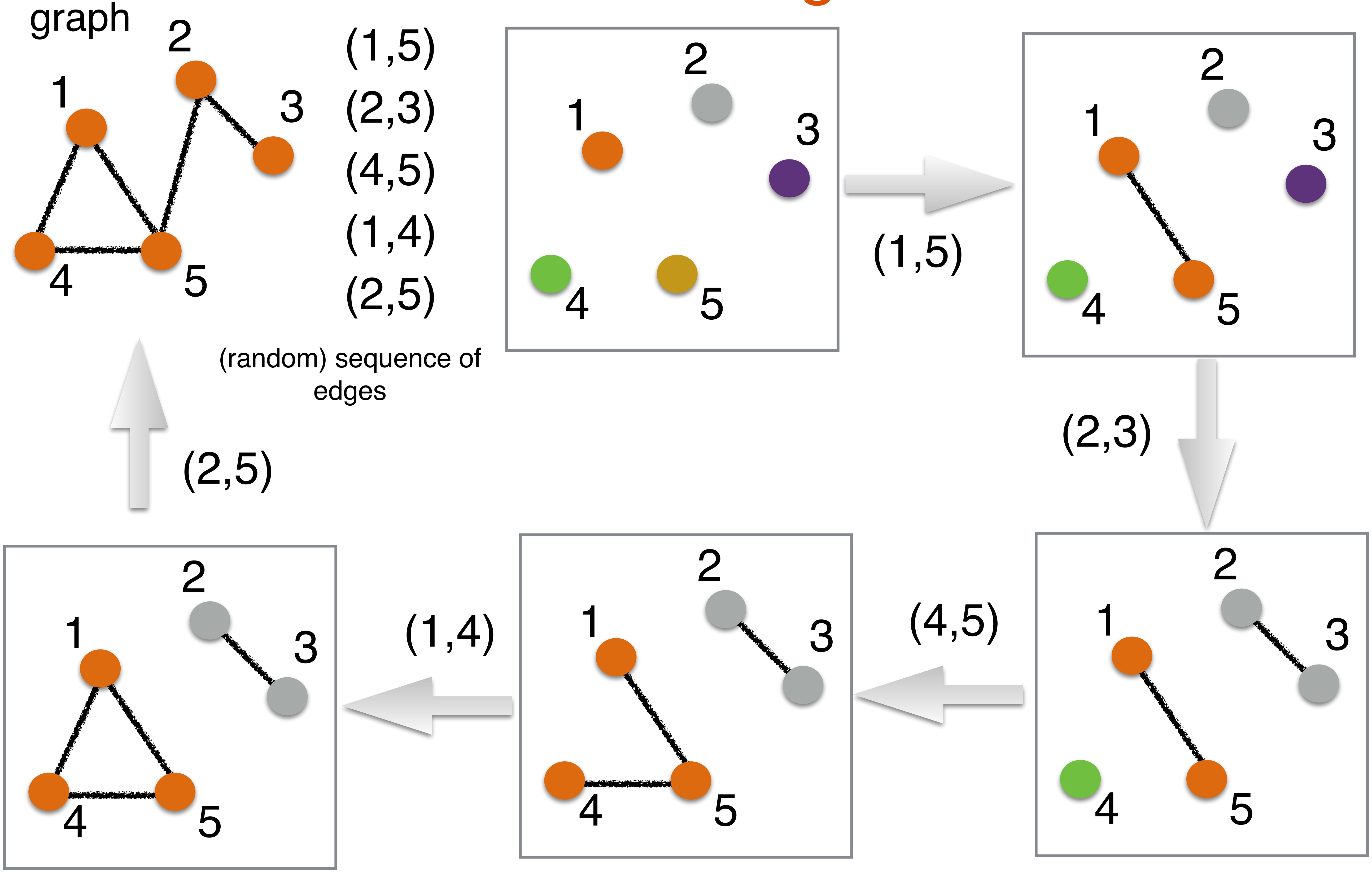
Results on a ER model with size N = 1000 and average degree k = 5.
Results are averaged over T = 100 realizations of the model.



each point in the diagram requires Nk/2 T elementary operations

each value of p can be considered independently

Newman-Ziff algorithm



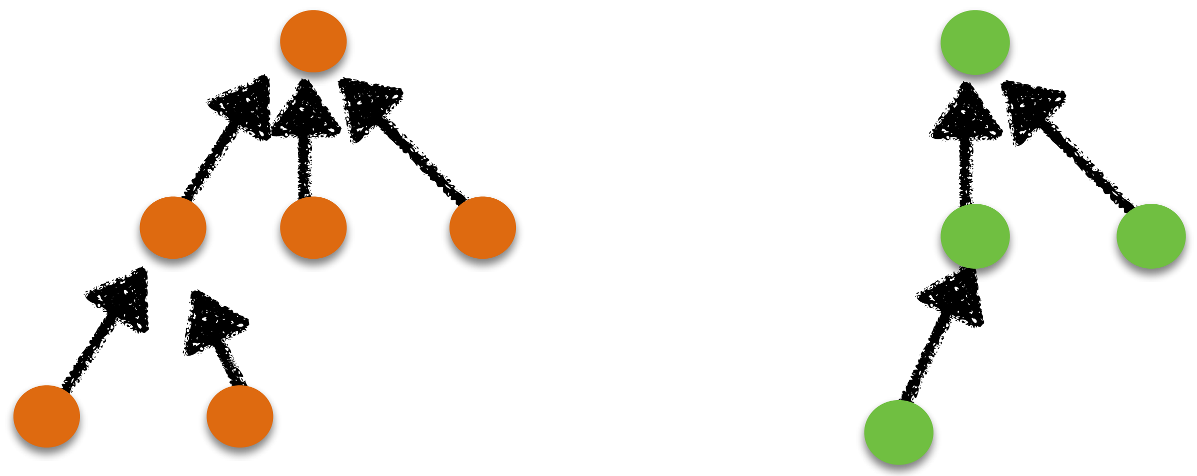
MEJ Newman and RM Ziff, Phys. Rev. Lett. **85**, 4104 (2000)

MEJ Newman and RM Ziff, Phys. Rev. E **64**, 016706 (2001)

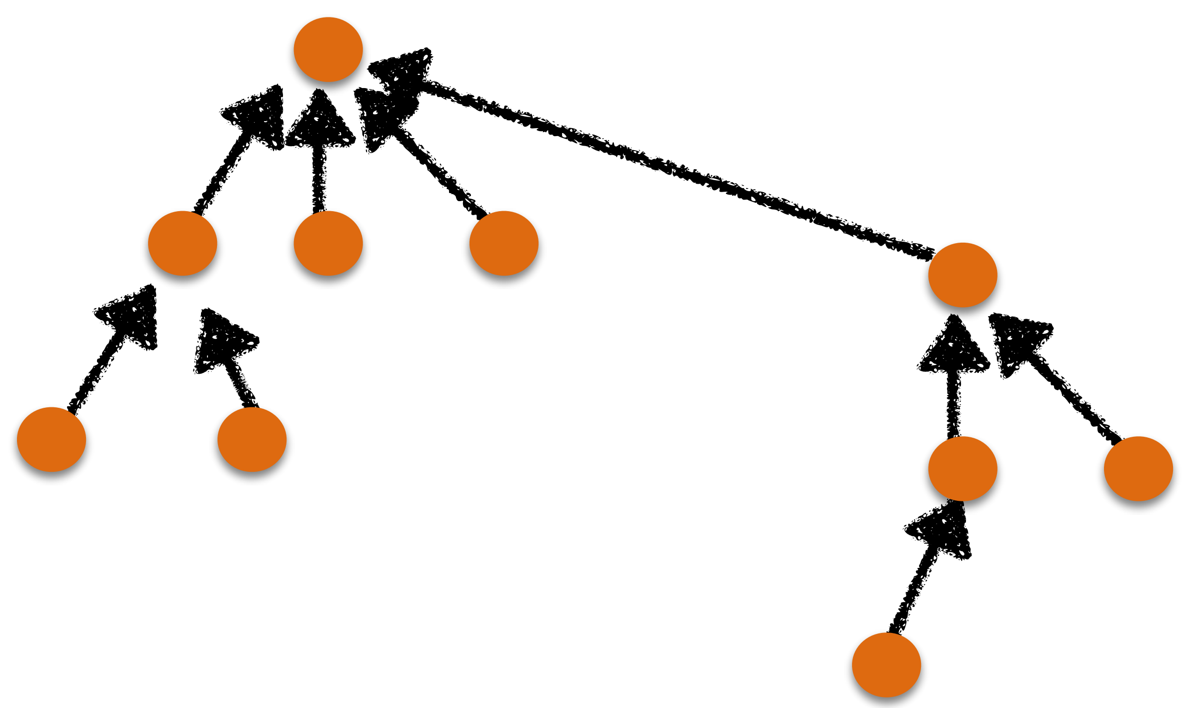
Newman-Ziff algorithm

each cluster is represented by a tree

the “color” of a cluster is given by the label of its root node



the merger of two clusters is obtained by making the root of the larger tree become the root of the smaller tree too



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Newman-Ziff algorithm

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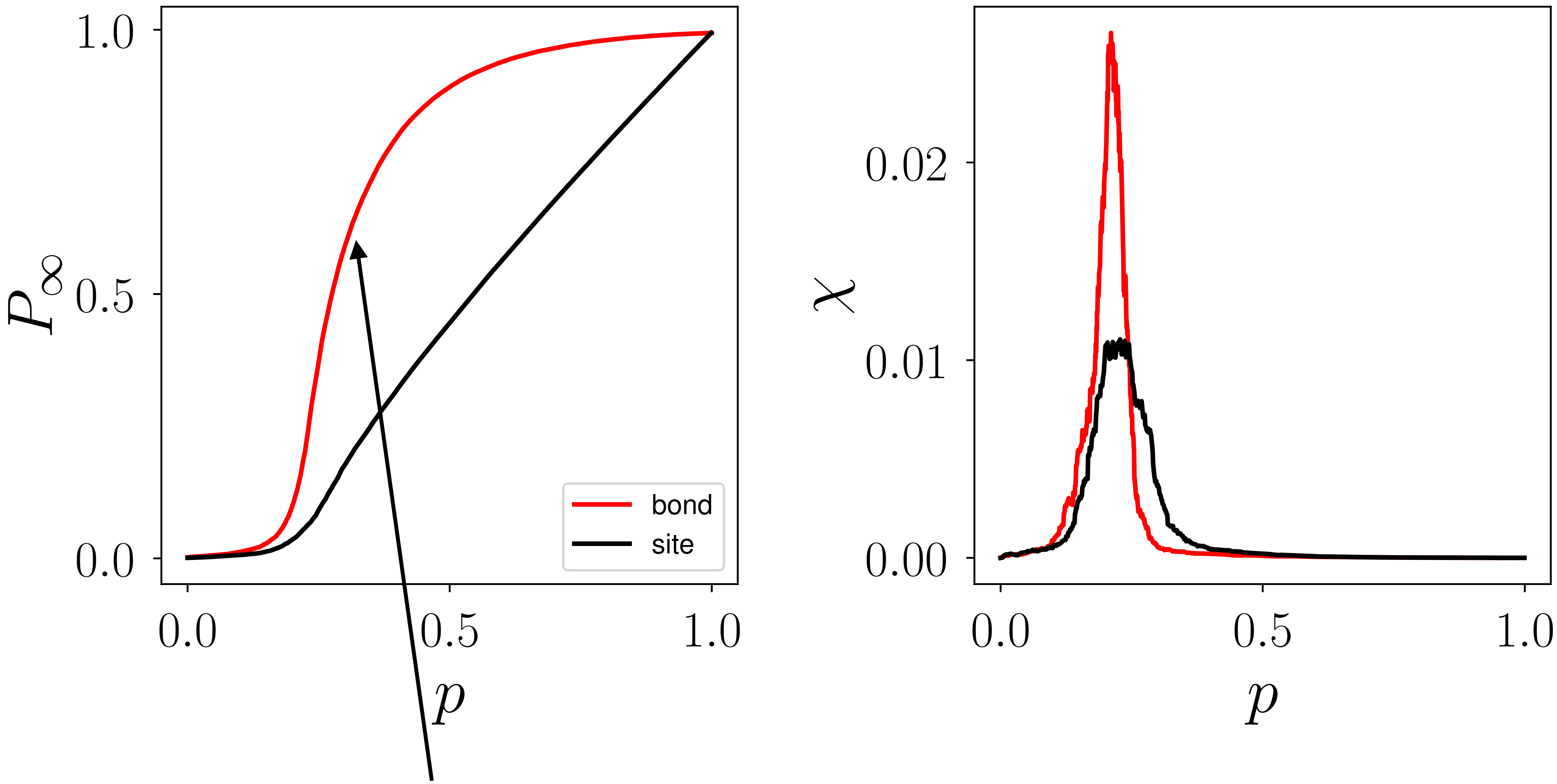

the p-th point in the diagram requires p Nk/2 T elementary operations

the diagram must be constructed from p=0 until a desired maximum value of p

Newman-Ziff algorithm

microcanonical ensemble

$$P_\infty(e)$$

the NZ algorithm natively allows us to estimate the value of an observable for a given number e of microscopic elements present in the network

canonical ensemble

$$P_\infty(p)$$

value of an observable for a given value of the occupation probability

$$P_\infty(p) = \sum_{e=0}^{E} P_\infty(e) \; \text{Binom.}(e|E,p)$$

$$\text{Binom.}(e|E,p) = \binom{E}{e} p^e (1-p)^{E-e}$$

MEJ Newman and RM Ziff, Phys. Rev. Lett. **85**, 4104 (2000)

MEJ Newman and RM Ziff, Phys. Rev. E **64**, 016706 (2001)

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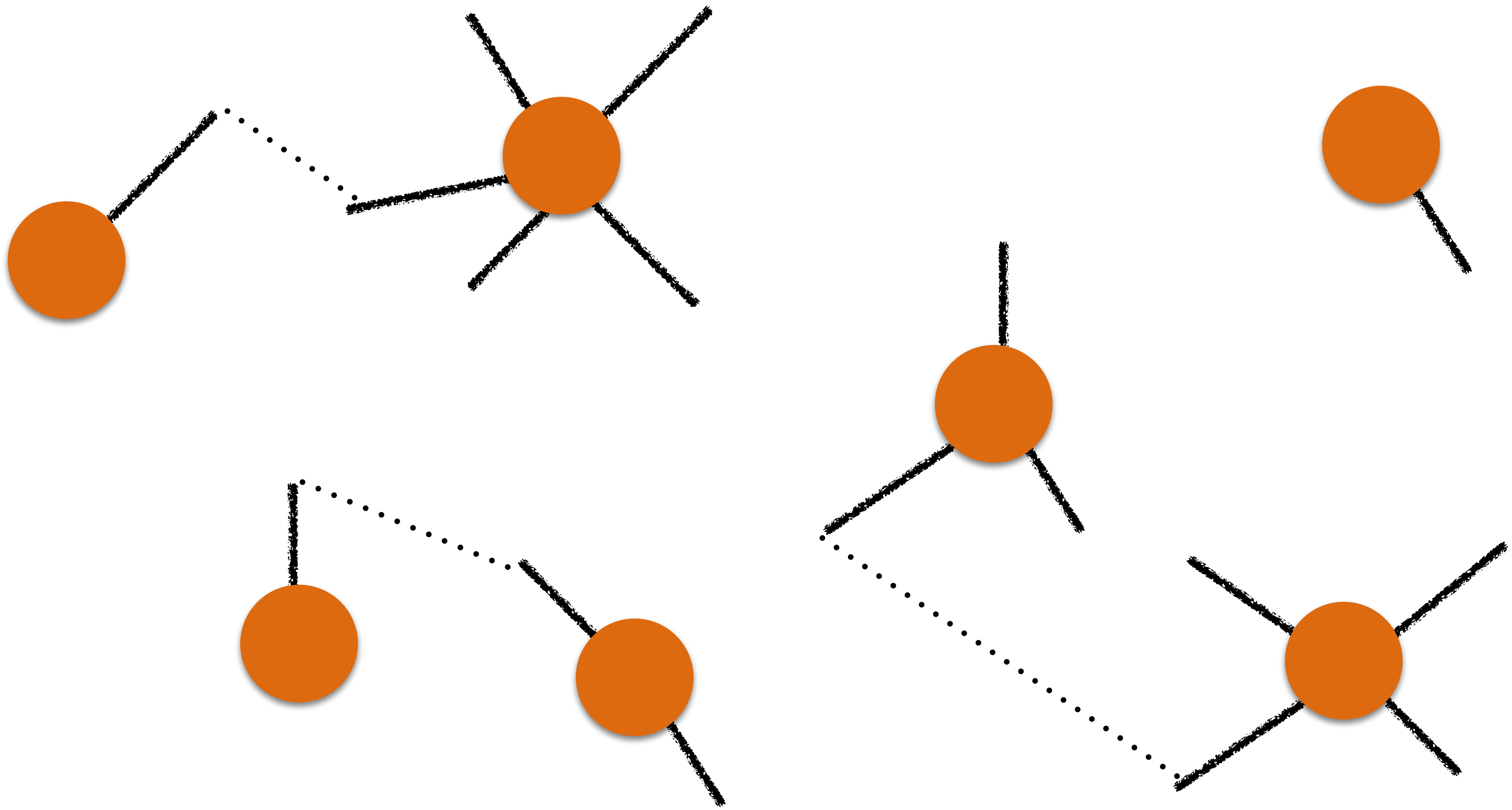
Percolation on random networks



configuration model: ensemble of random uncorrelated networks with prescribed degree sequence

Molloy, M., and Reed, B. *Random structures & algorithms* 6.2-3 (1995): 161-180.

Callaway, D. S., Newman, M. E., Strogatz, S. H. & Watts, D. J. *Phys. Rev. Lett.* 85, 5468 (2000).

Cohen, R., Erez, K., Ben-Avraham, D. & Havlin, *Phys. Rev. Lett.* 85, 4626 (2000).

Cohen, R., Ben-Avraham, D. & Havlin, *Phys. Rev. E* 66, 036113 (2002).

Classical results for percolation in networks

bond and site percolation on random uncorrelated networks

percolation threshold

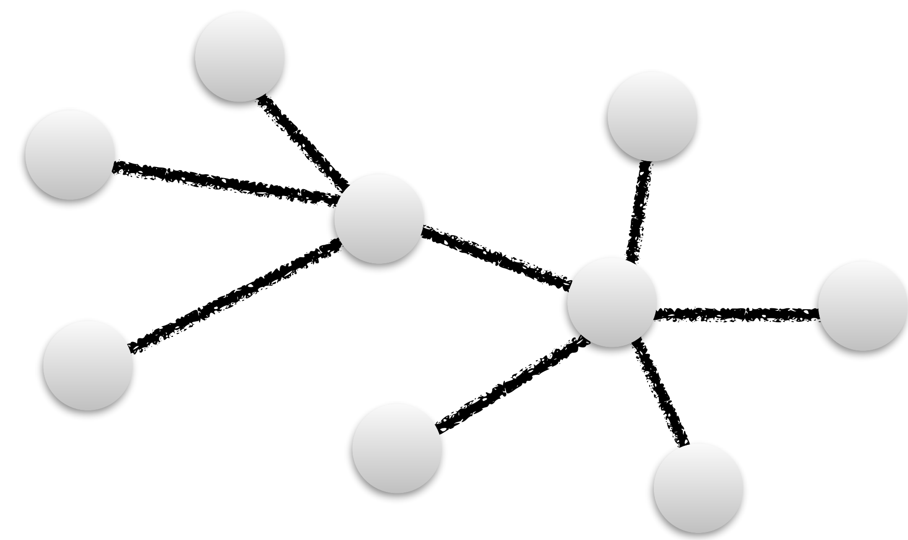
$$p_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$

critical exponents for SF networks
with degree distribution $P(k) \sim k^{-\gamma}$

$$\nu = \begin{cases} 2/(3-\gamma) & , \text{ if } 2 < \gamma < 3 \\ (\gamma-1)/(\gamma-3) & , \text{ if } 3 < \gamma \leq 4 \\ 3 & , \text{ if } \gamma \geq 4 \end{cases}$$

$$\beta = \begin{cases} 1/(3-\gamma) & , \text{ if } 2 < \gamma < 3 \\ 1/(\gamma-3) & , \text{ if } 3 < \gamma \leq 4 \\ 1 & , \text{ if } \gamma \geq 4 \end{cases}$$

approximations



locally tree-like ansazt

$$A_{i,j} \sim k_i \, k_j$$

annealed network approximation

Callaway, D. S., Newman, M. E., Strogatz, S. H. & Watts, D. J. *Phys. Rev. Lett.* 85, 5468 (2000).
Cohen, R., Erez, K., Ben-Avraham, D. & Havlin, *Phys. Rev. Lett.* 85, 4626 (2000).
Cohen, R., Ben-Avraham, D. & Havlin, *Phys. Rev. E* 66, 036113 (2002).

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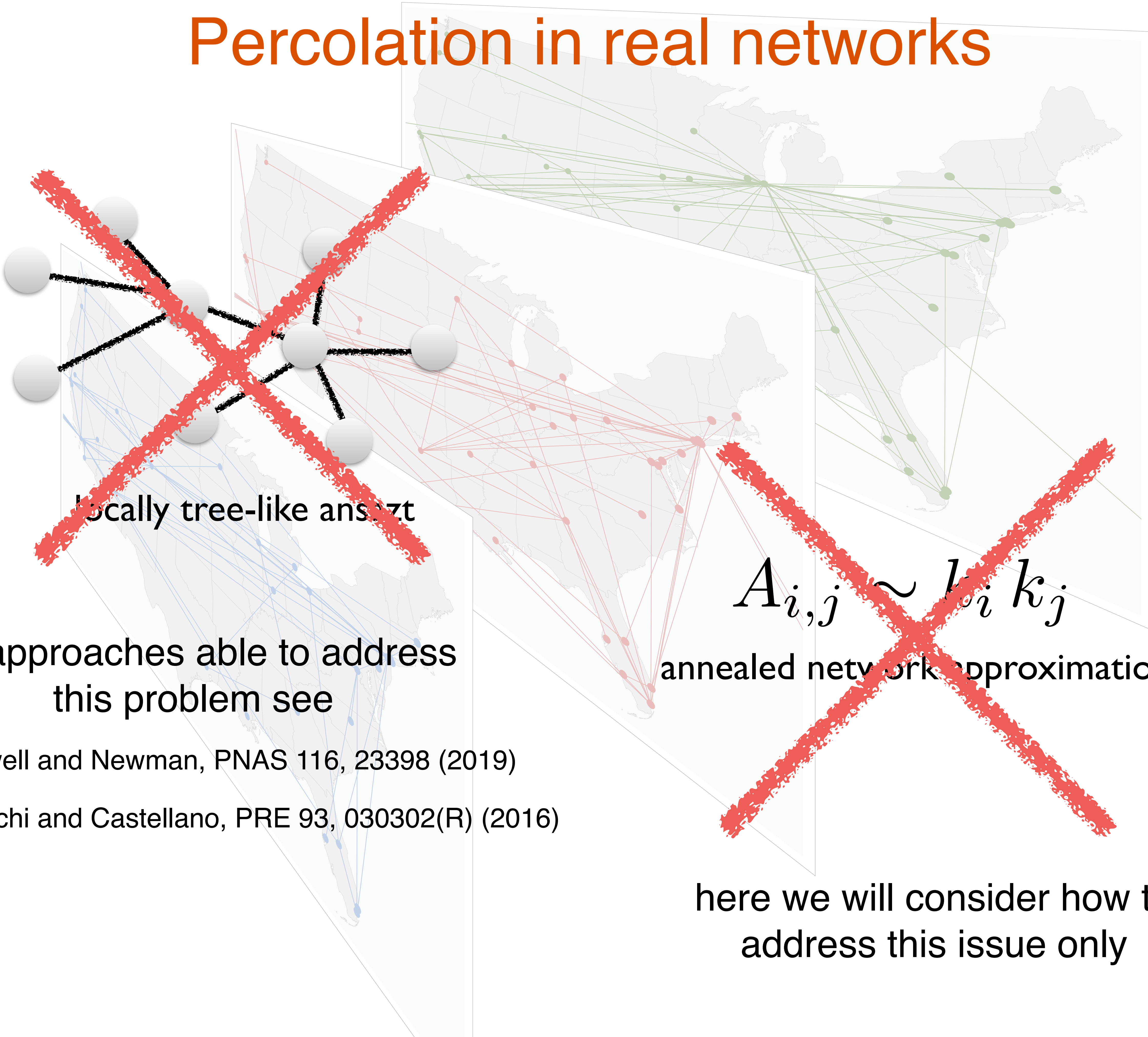
```

Percolation in real networks



locally tree-like ansatz

$$A_{i,j} \sim k_i\, k_j$$

annealed network approximation

for approaches able to address this problem see

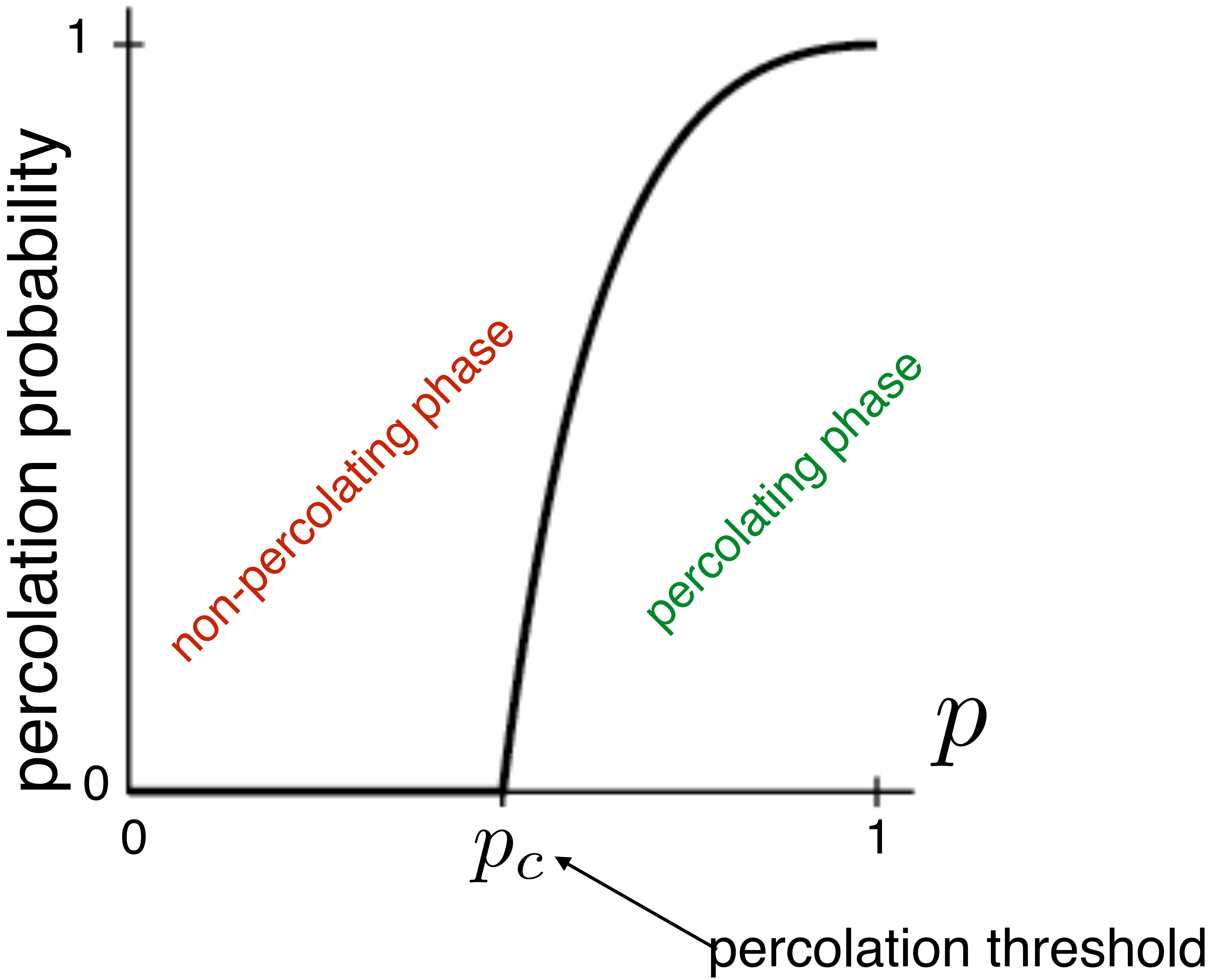
Cantwell and Newman, PNAS 116, 23398 (2019)

Radicchi and Castellano, PRE 93, 030302(R) (2016)

here we will consider how to address this issue only

Percolation transition

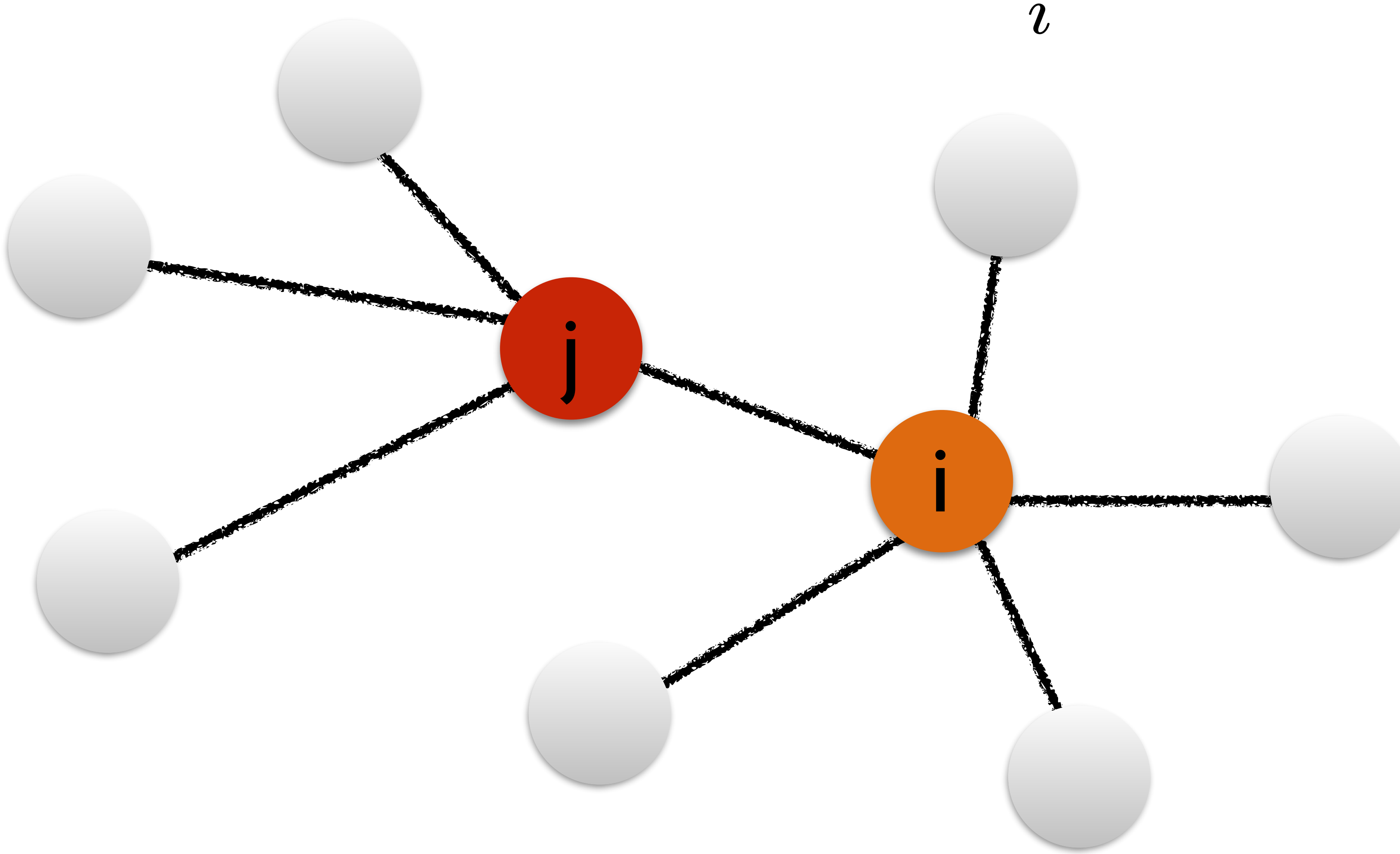
a true percolation transition occurs only in infinite-size networks!



Site percolation in real networks

s_i = prob. node i in the GC

$$P_\infty = \frac{1}{N}\sum_i s_i$$ = percolation strength



$$s_i = p\left[1 - \prod_{j\in\mathcal{N}_i}(1 - s_j)\right]$$

set of neighbors of node i

Son,S.-W.,Bizhani,G.,Christensen,C.,Grassberger,P. & Paczuski,M. *EPL (Europhysics Letters)* 97, 16006 (2012).

Site percolation in real networks

$$s_i = p\,[1 - \prod_{j \in \mathcal{N}_i} (1 - s_j)\,]$$

$$q_i = \ln(1 - s_i/p)$$
$$u_i = \ln(1 - s_i)$$

↓

$$\vec{q} = A\,\vec{u}$$

truncated Taylor expansion

$$\ln(1 - x) \simeq -x$$

↓

eigenvalue/eigenvector problem

$$\vec{s} = pA\vec{s}$$

↓

Perron-Frobenius theorem

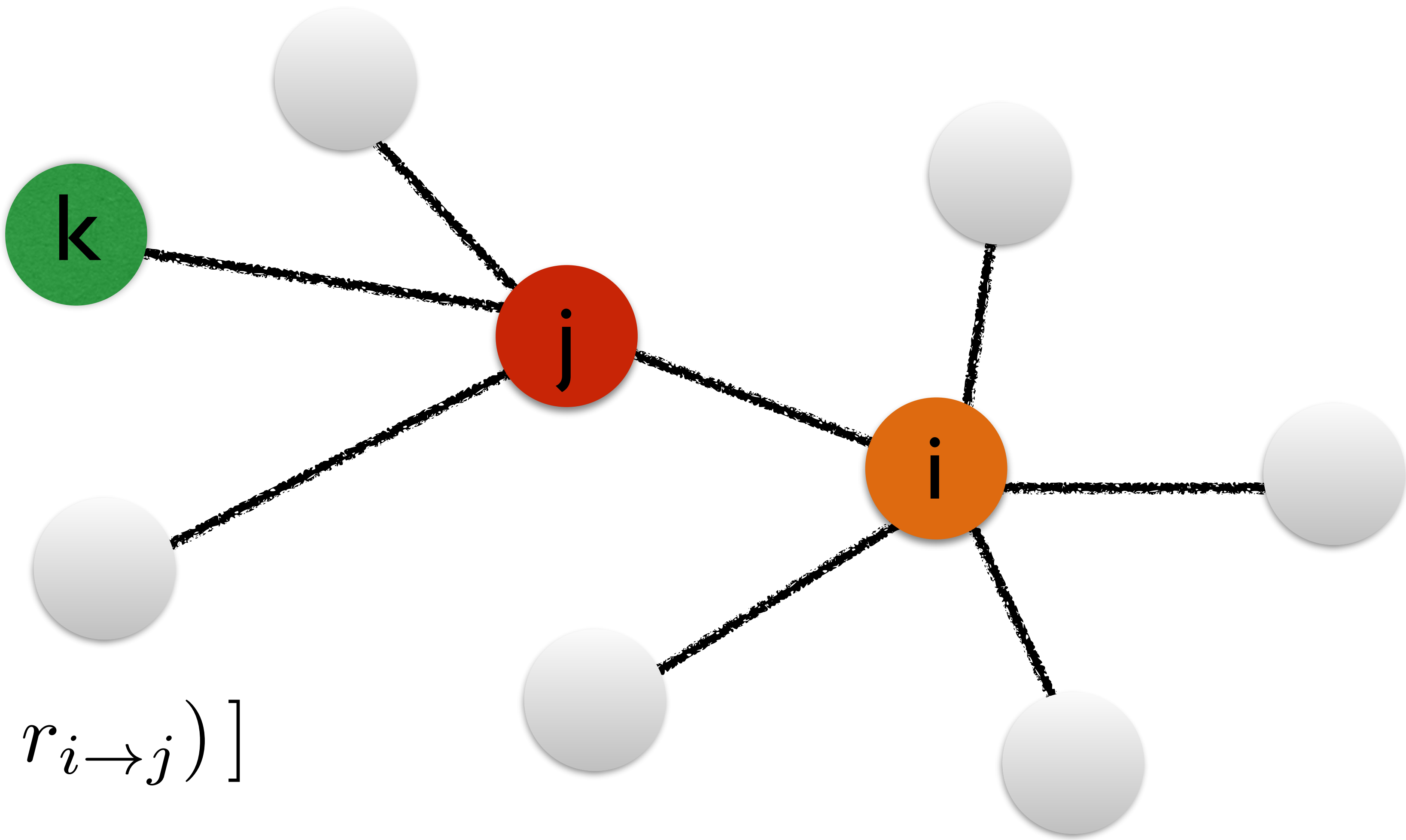
$$p_c = \frac{1}{\lambda}$$

Bollobas, B., Borgs, C., Chayes, J., Riordan, O. *et al.* *The Annals of Probability* 38, 150–183 (2010).

Site percolation in real networks

s_i = prob. node i in the GC

$r_{i\to j}$ = prob. node j in the GC disregarding node i



$$s_i = p\,[\,1 - \prod_{j\in\mathcal{N}_i} (1 - r_{i\to j})\,]$$

$$r_{i\to j} = p\,[1 - \prod_{k\in\mathcal{N}_j\setminus\{i\}} (1 - r_{j\to k})\,]$$

Karrer, B., Newman, M. E. J. & Zdeborova, L. *Phys. Rev. Lett.* 113, 208702 (2014).

Hamilton, K. E. & Pryadko, L. P. *Phys. Rev. Lett.* 113, 208701 (2014).

Site percolation in real networks

$$r_{i\to j} = p\left[1 - \prod_{k\in\mathcal{N}_j\setminus\{i\}} (1 - r_{j\to k})\right]$$

$$w_{i\to j} = \ln(1 - r_{i\to j})$$
$$z_{i\to j} = \ln(1 - r_{i\to j}/p)$$
$$M_{i\to j,k\to\ell} = \delta_{j,k}(1 - \delta_{i,\ell})$$

↓

$$\vec{z} = M\,\vec{w}$$

↓

truncated Taylor expansion

$$\ln(1 - x) \simeq -x$$

eigenvalue/eigenvector problem

$$\vec{r} = p\,M\,\vec{r}$$

↓

Perron-Frobenius theorem

$$p_c = \frac{1}{\mu}$$

Karrer, B., Newman, M. E. J. & Zdeborova, L. *Phys. Rev. Lett.* 113, 208702 (2014).

Hamilton, K. E. & Pryadko, L. P. *Phys. Rev. Lett.* 113, 208701 (2014).

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Libraries used in this notebook

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import random
```
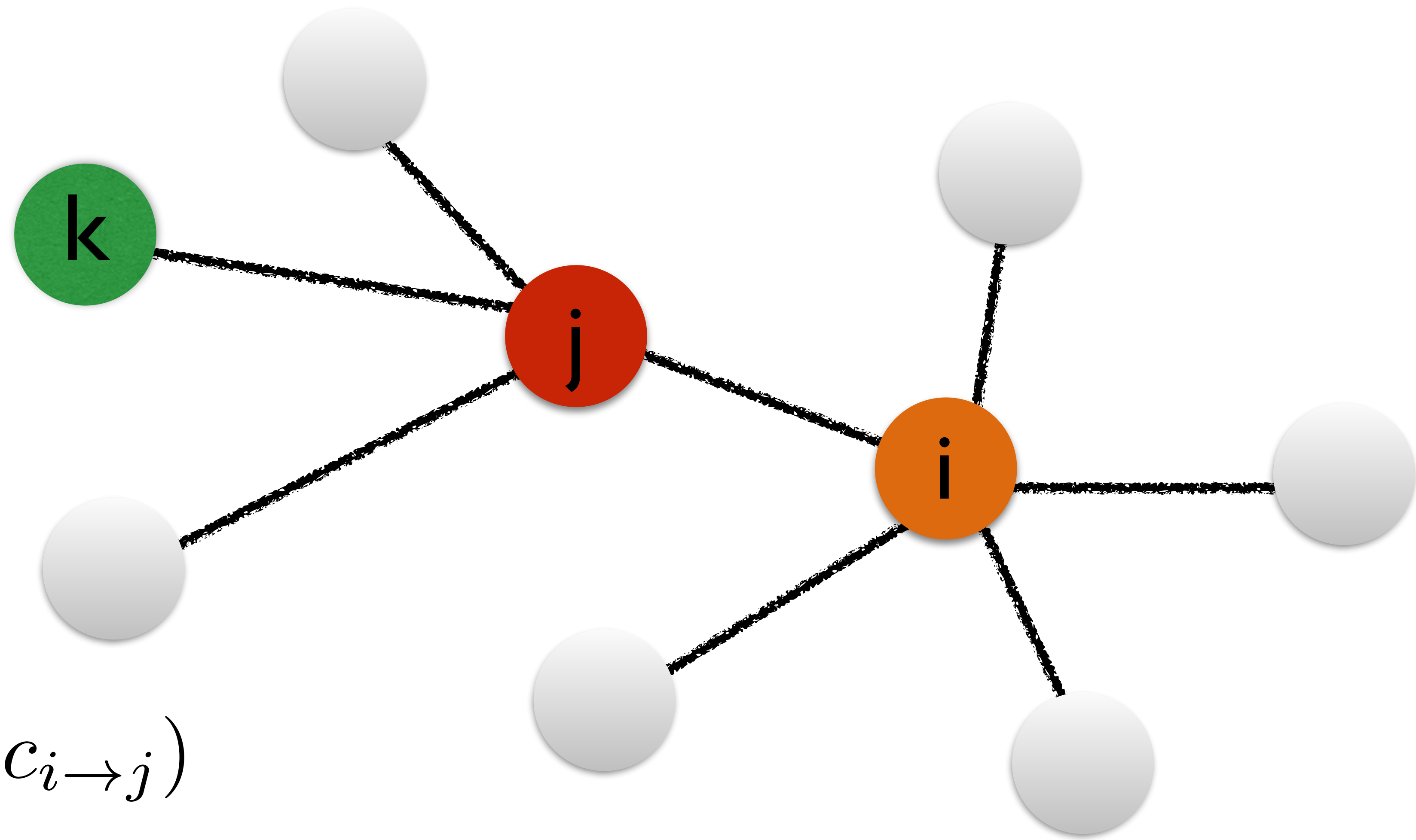

Bond percolation in real networks

b_i = prob. node i in the GC

$c_{i\to j}$ = prob. node j in the GC disregarding node i



$$b_i = 1 - \prod_{j \in \mathcal{N}_i} (1 - p\, c_{i\to j})$$

$$c_{i\to j} = 1 - \prod_{k \in \mathcal{N}_j \setminus \{i\}} (1 - p\, c_{j\to k})$$

Karrer, B., Newman, M. E. J. & Zdeborova, L. *Phys. Rev. Lett.* 113, 208702 (2014).

Hamilton, K. E. & Pryadko, L. P. *Phys. Rev. Lett.* 113, 208701 (2014).

Bond and site percolation in real networks

site percolation

$$s_i = p\,[\,1 - \prod_{j \in \mathcal{N}_i} (1 - r_{i \to j})\,]$$

$$r_{i \to j} = p\,[1 - \prod_{k \in \mathcal{N}_j \setminus \{i\}} (1 - r_{j \to k})\,]$$

bond percolation

$$b_i = 1 - \prod_{j \in \mathcal{N}_i} (1 - p\, c_{i \to j})$$

$$c_{i \to j} = 1 - \prod_{k \in \mathcal{N}_j \setminus \{i\}} (1 - p\, c_{j \to k})$$

They are the same equations with

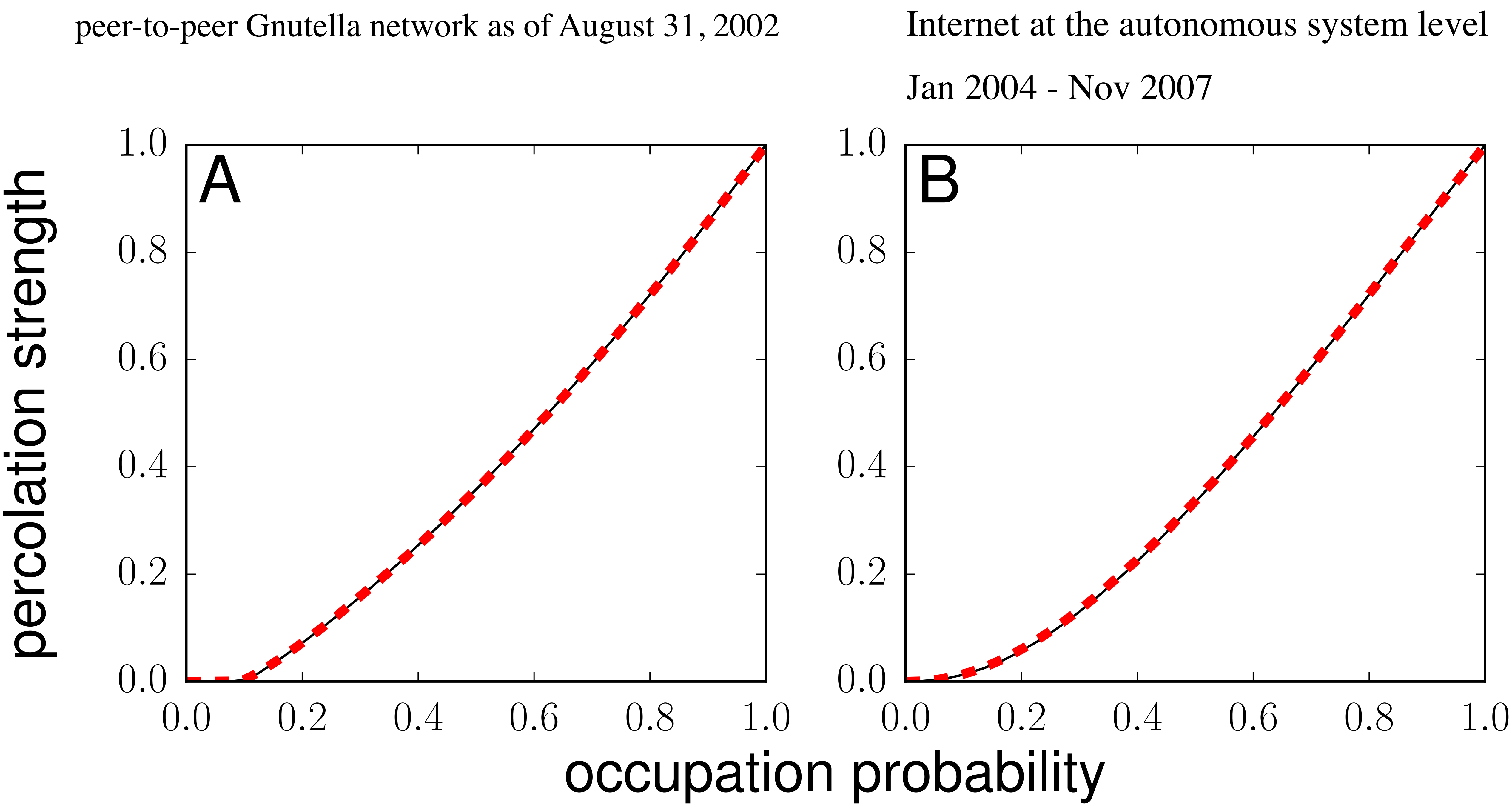
$$s_i = p\, b_i \qquad r_{i \to j} = p\, c_{i \to j}$$

$$P_\infty^{(site)} = p P_\infty^{(bond)}$$

Radicchi & Castellano, Nat. Commun. **6**, 10196 (2015)

Bond and site percolation in real networks

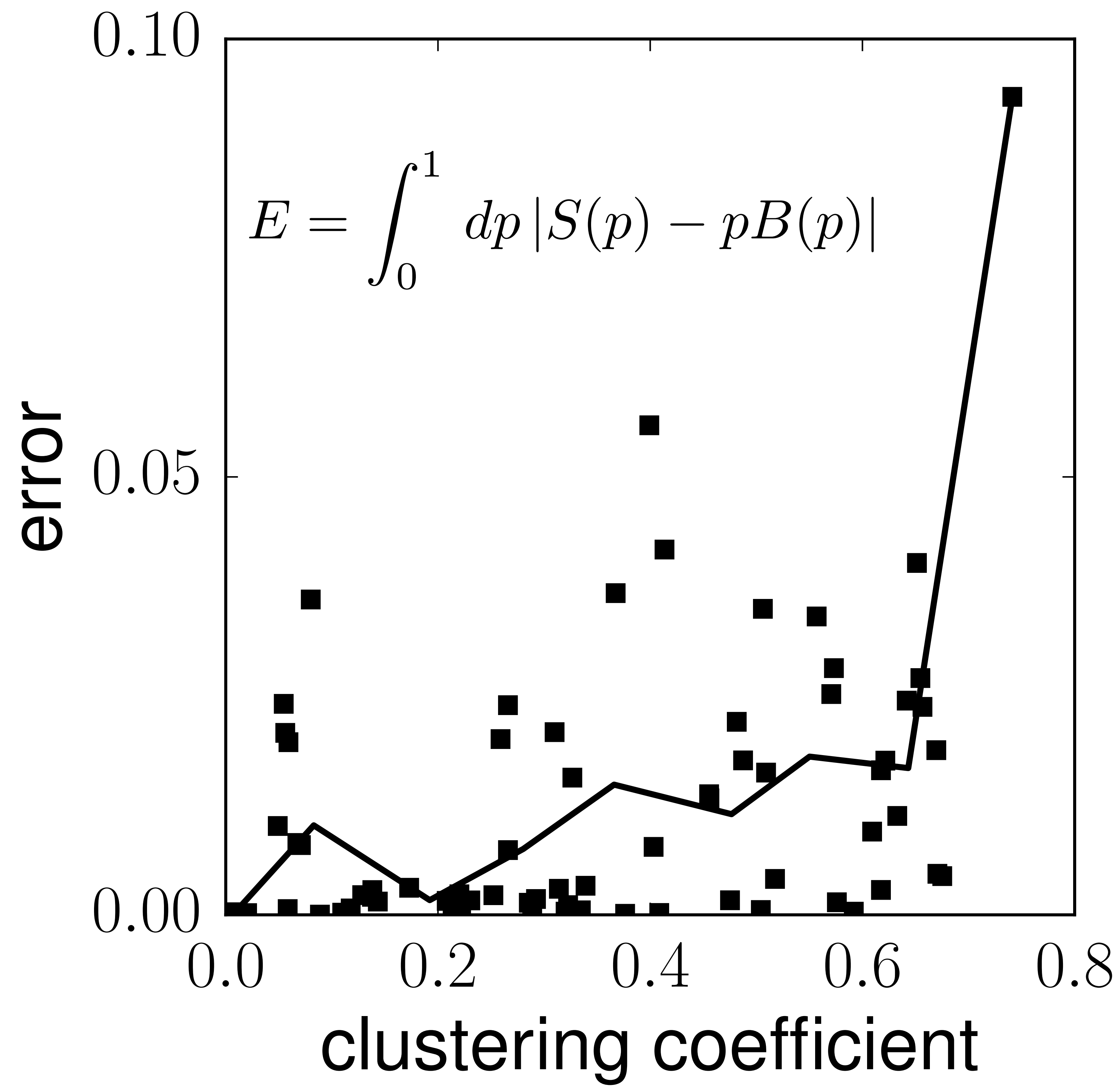
results of numerical simulations for real networks



Radicchi & Castellano, Nat. Commun. **6**, 10196 (2015)

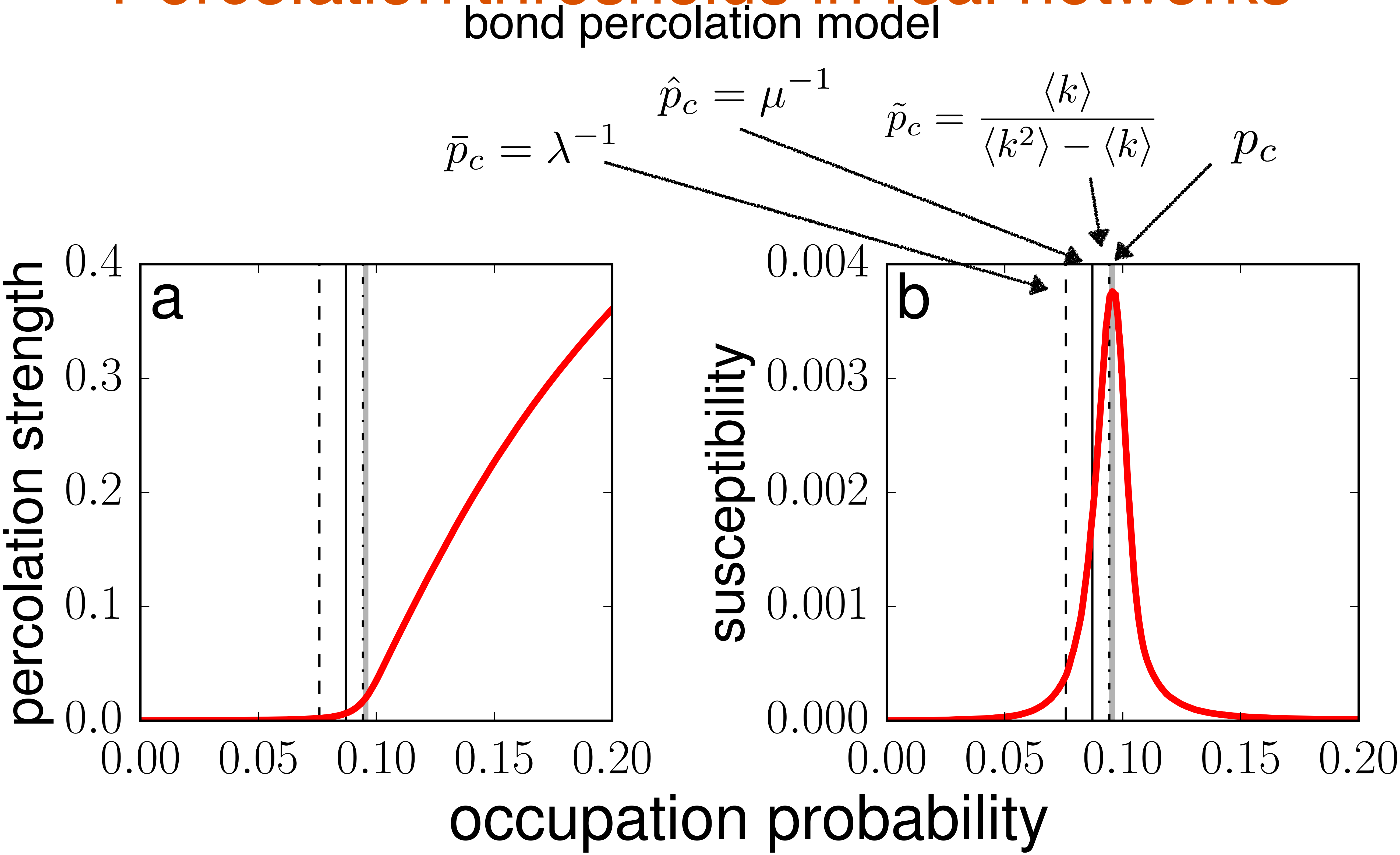
Bond and site percolation in real networks

results of numerical simulations for 109 real networks



Radicchi & Castellano, Nat. Commun. **6**, 10196 (2015)

Percolation thresholds in real networks

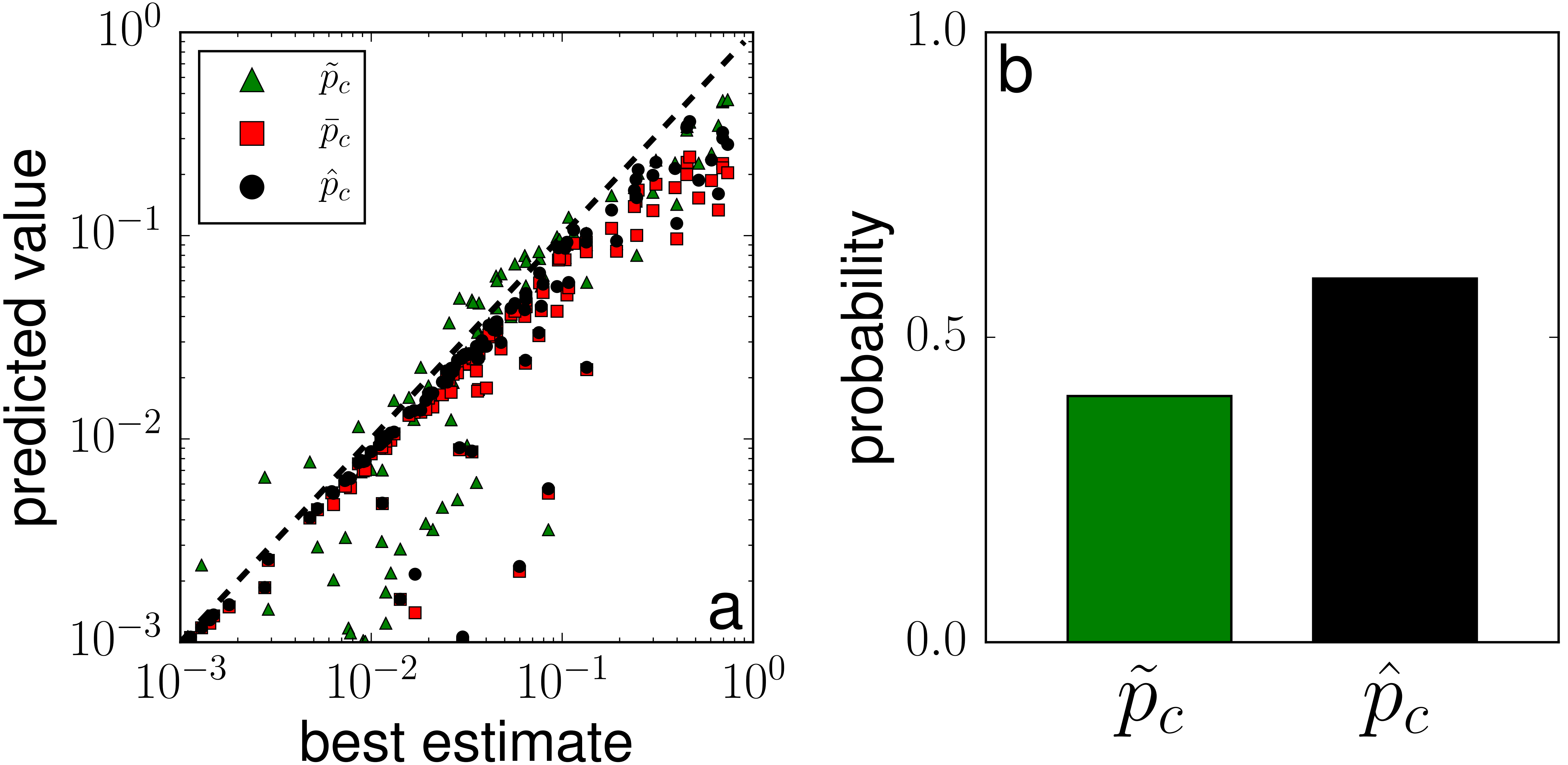


peer-to-peer Gnutella network as of August 31, 2002

F. Radicchi, Predicting percolation thresholds in networks, Phys. Rev. E 91, 010801(R) (2015)

Percolation thresholds in real networks

based on the analysis of 109 real networks



F. Radicchi, Predicting percolation thresholds in networks, Phys. Rev. E 91, 010801(R) (2015)

Breaking of site-bond percolation universality

in locally tree-like graphs with null percolation threshold

$$P_\infty^{(site)} \sim (p - p_c)^{\beta_s}$$

$$P_\infty^{(bond)} \sim (p - p_c)^{\beta_b}$$

$$P_\infty^{(site)} = p P_\infty^{(bond)}$$

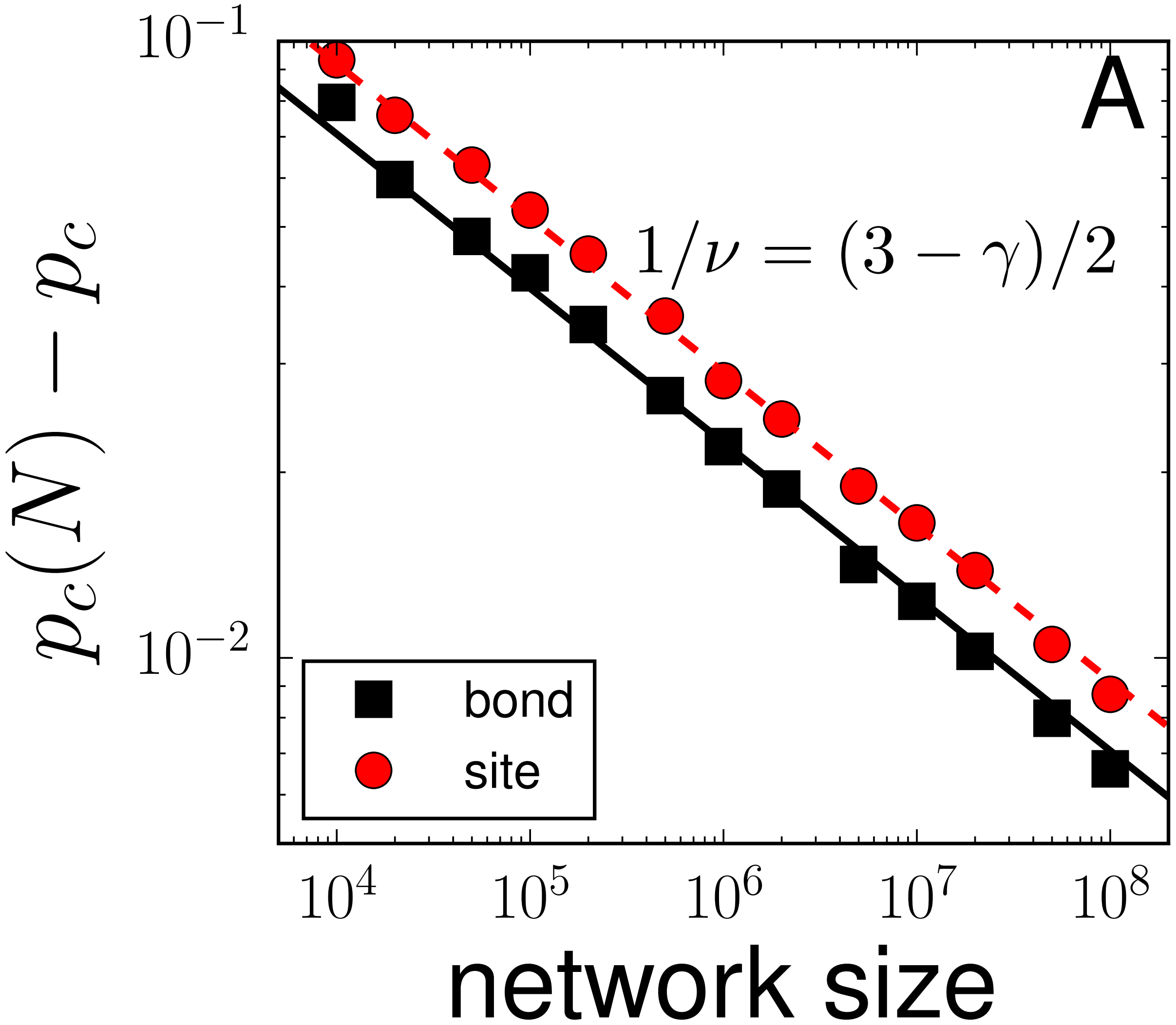
if $p_c > 0$ **then** $\beta_s = \beta_b$

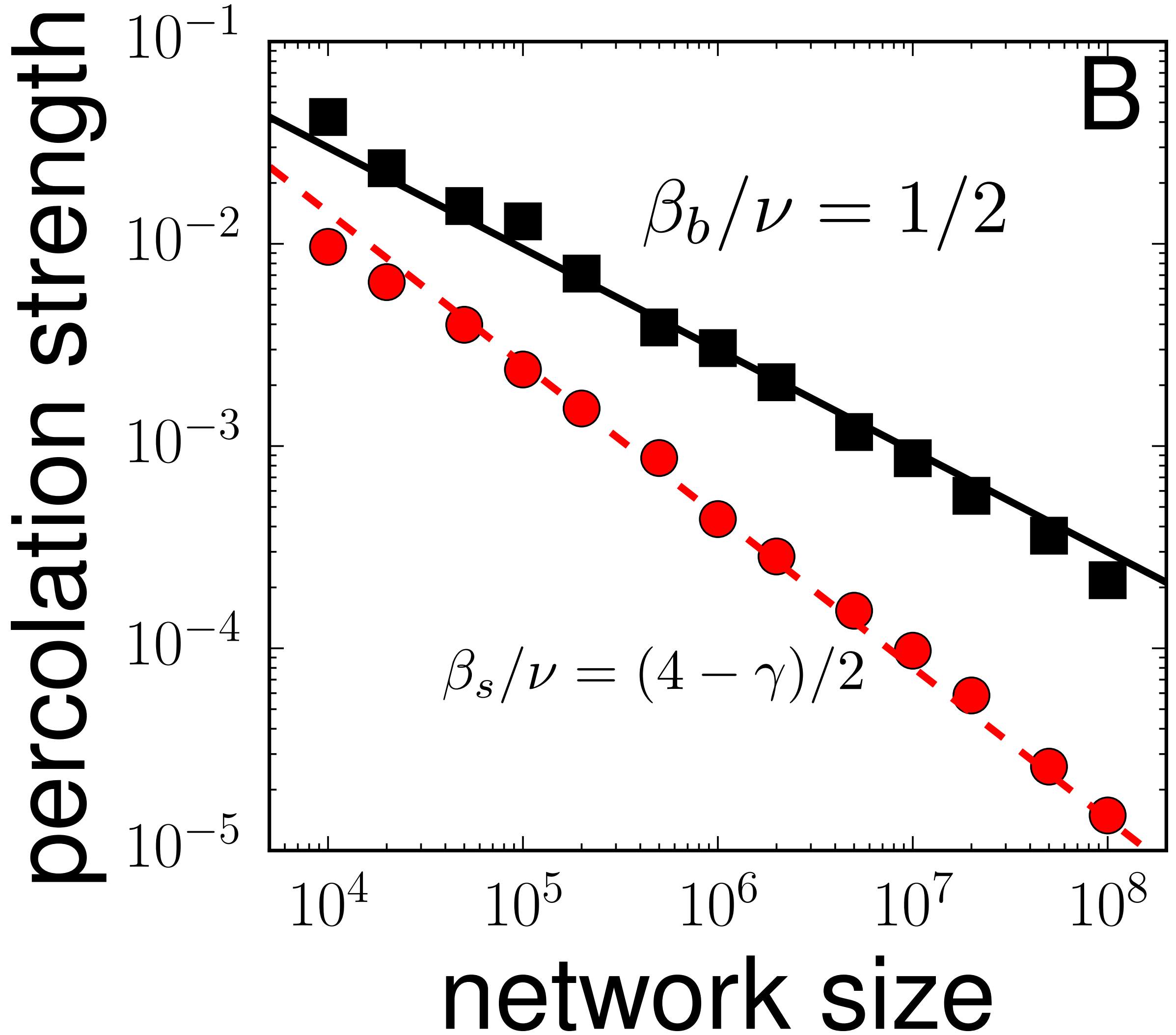
if $p_c = 0$ **then** $\beta_s = \beta_b + 1$

Radicchi & Castellano, Nat. Commun. **6**, 10196 (2015)

Breaking of site-bond percolation universality

SF graphs $\gamma = 5/2$





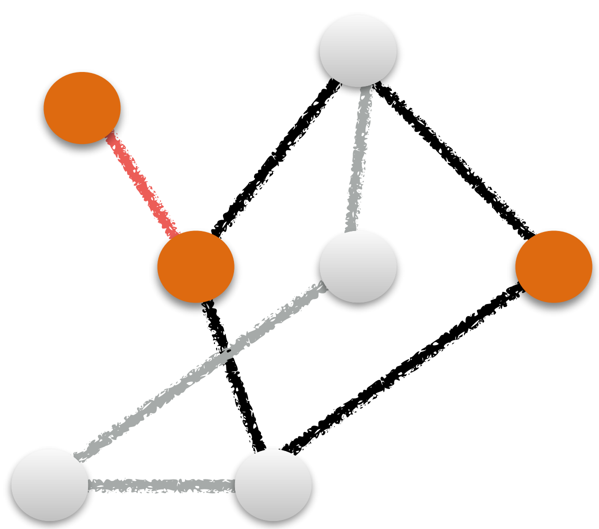
Cohen R, Ben-Avraham D, Havlin S (2002) Percolation critical exponents in scale-free networks. Physical Review E 66(3):036113.

Radicchi & Castellano, Nat. Commun. **6**, 10196 (2015)

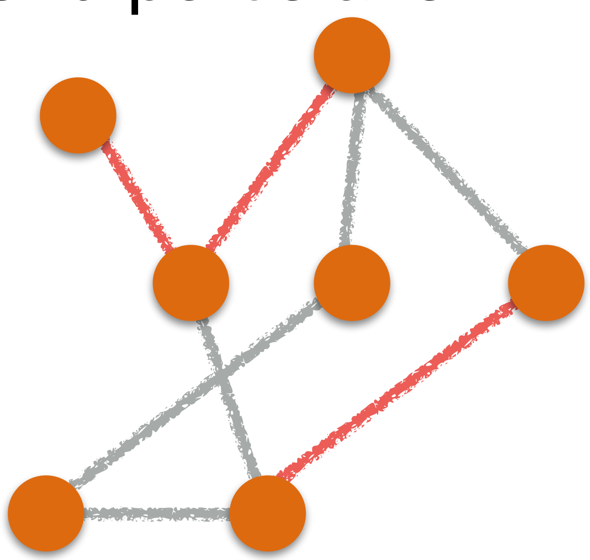
Percolation models in networks

microscopic elements to be added/removed

site percolation



bond percolation



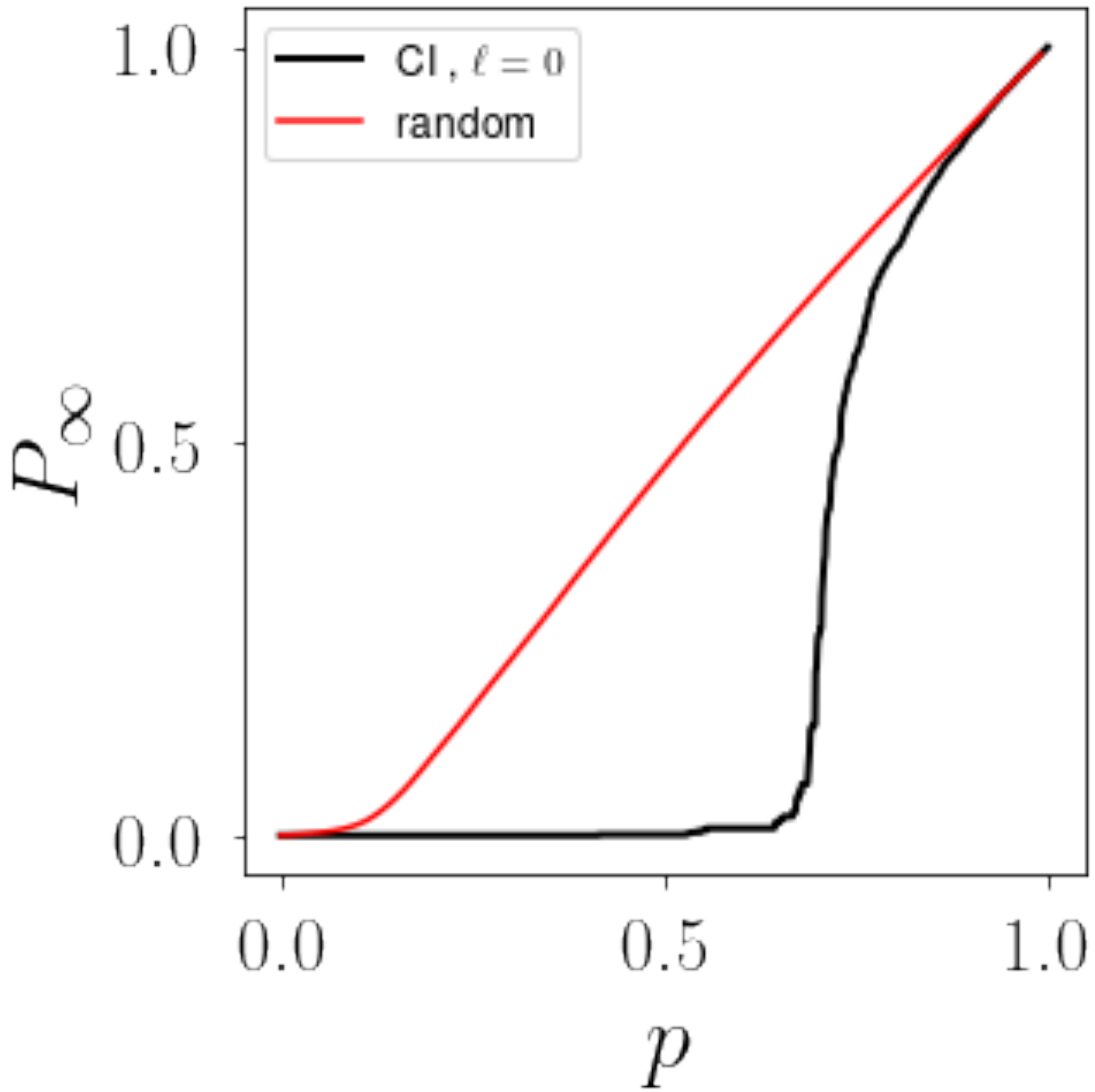
protocols for the addition/removal of microscopic elements

protocol	model	
random	ordinary	so far, we only consider this model
degree-based	targeted attacks	Reka Albert, Hawoong Jeong, Albert-Laszlo Barabasi Nature 406, 378-382 (2000)
Achlioptas decision process	explosive	DIMITRIS ACHLIOPTAS, RAISSA M. D'SOUZA, AND JOEL SPENCER, Science 323, 1453 (2009)
optimized deletion	optimal	Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)

Targeted attacks

the occupation probability is a function of topology of the underlying network

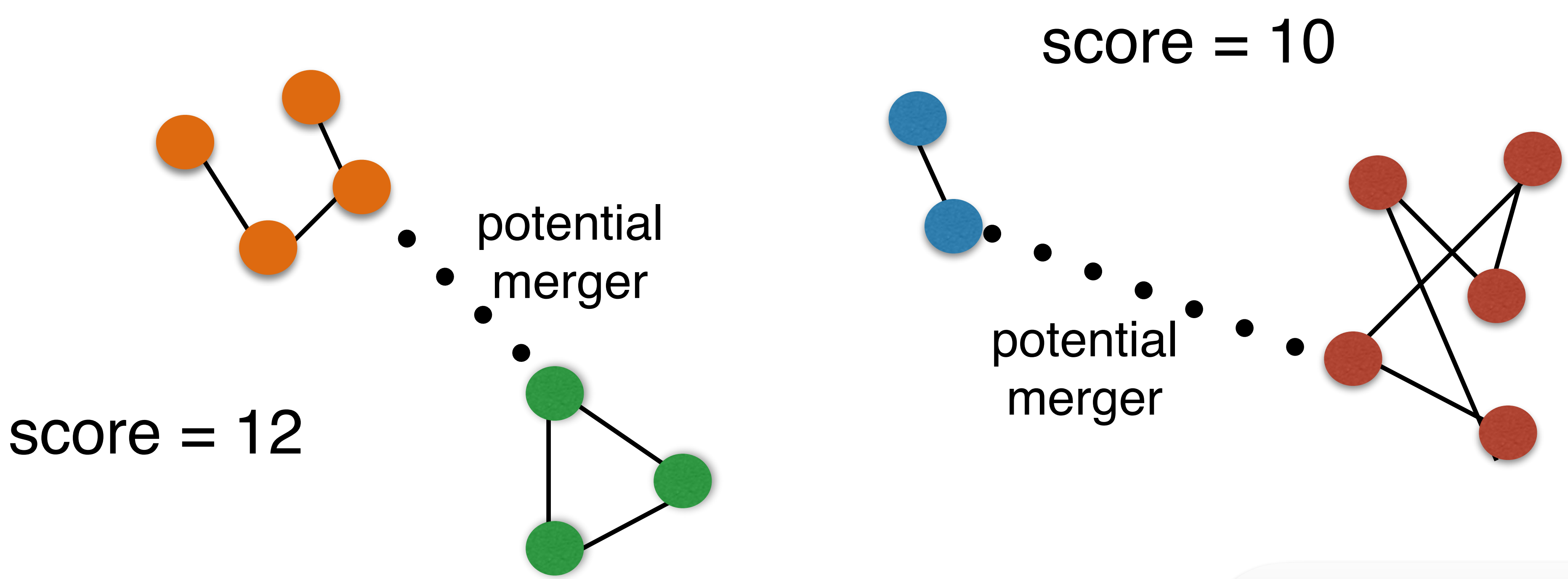
Results on a configuration model with size N = 1000 and power-law degree distribution with exponent gamma = 2.1.



Reka Albert, Hawoong Jeong, Albert-Laszlo Barabasi, Nature 406, 378-382 (2000)

Explosive percolation

postponing the emergence of the macroscopic cluster



m edges are randomly selected, the one corresponding to the minimum of the product of the cluster sizes that is merging is added to the network

DIMITRIS ACHLIOPTAS, RAISSA M. D'SOUZA, AND JOEL SPENCER, Science 323, 1453 (2009)

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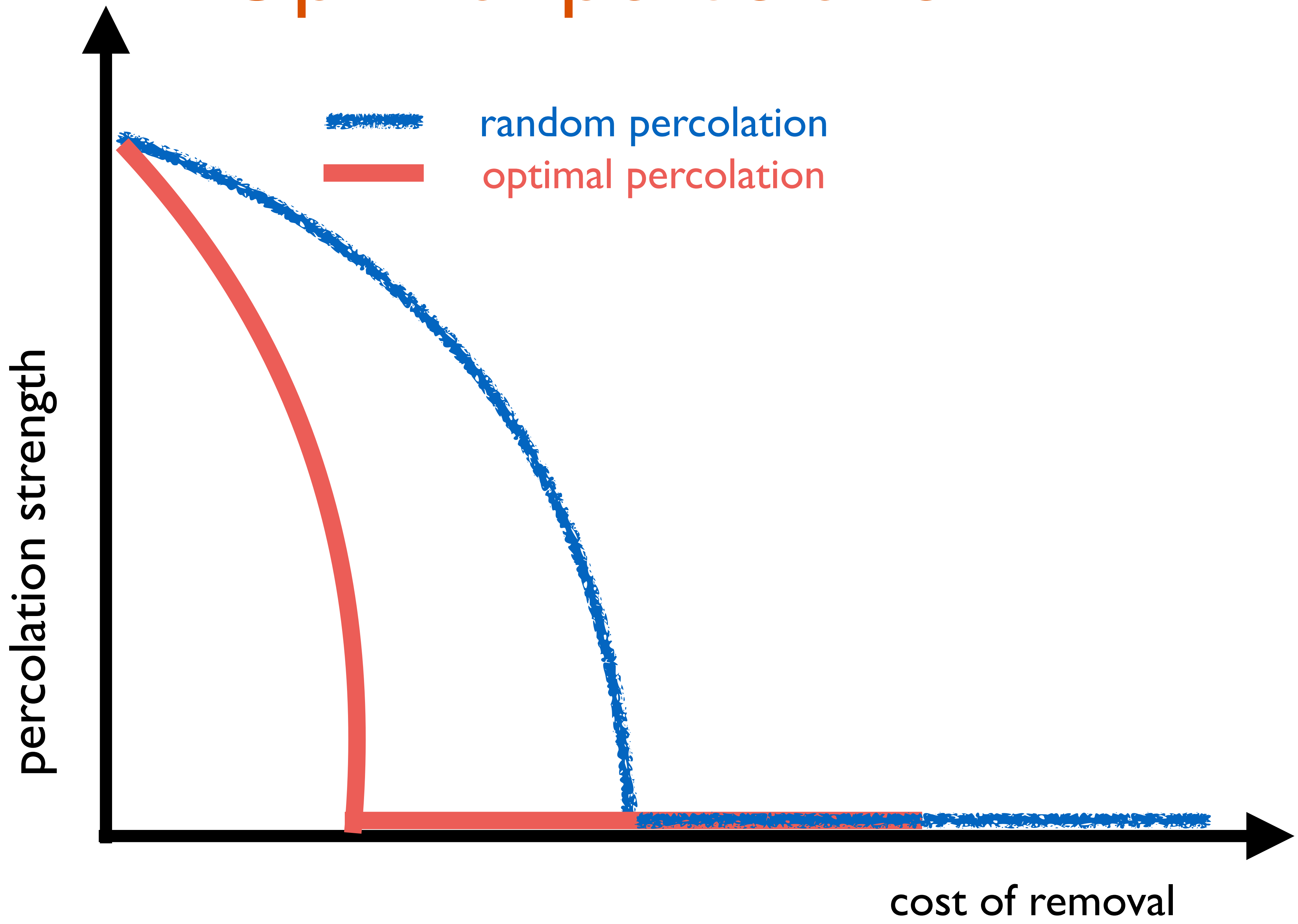
Optimal percolation



Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)

Xiao-Long Ren et al., PNAS 116, 6554 (2019)

Optimal percolation

Optimal structural set

$$\mathcal{S}_c = \arg \min_{\mathcal{S} | P_\infty(\mathcal{S}) \leq 1/\sqrt{N}} F(\mathcal{S})$$

Size of the largest component after removing the elements of the set S

conventional threshold value

Cost associated to the removal of the elements of the set S

$$F(\mathcal{S}_c)$$

Dismantling cost

Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)

Xiao-Long Ren, et al. , PNAS 116, 6554–6559 (2019)

Optimal percolation

Optimal structural set for a given budget

Size of the largest component after removing the elements of the set S

$$\mathcal{S}^*(C) = \arg \min_{\mathcal{S}|F(\mathcal{S})=C} P_\infty(\mathcal{S})$$

Cost associated to the removal of the elements of the set S

Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)

Xiao-Long Ren, et al. , PNAS 116, 6554–6559 (2019)

Optimal percolation

exact solution

Brute-force search

There are $\binom{N}{|\mathcal{S}|}$ possible sets of structural sets of size $|\mathcal{S}|$

in a network with N nodes

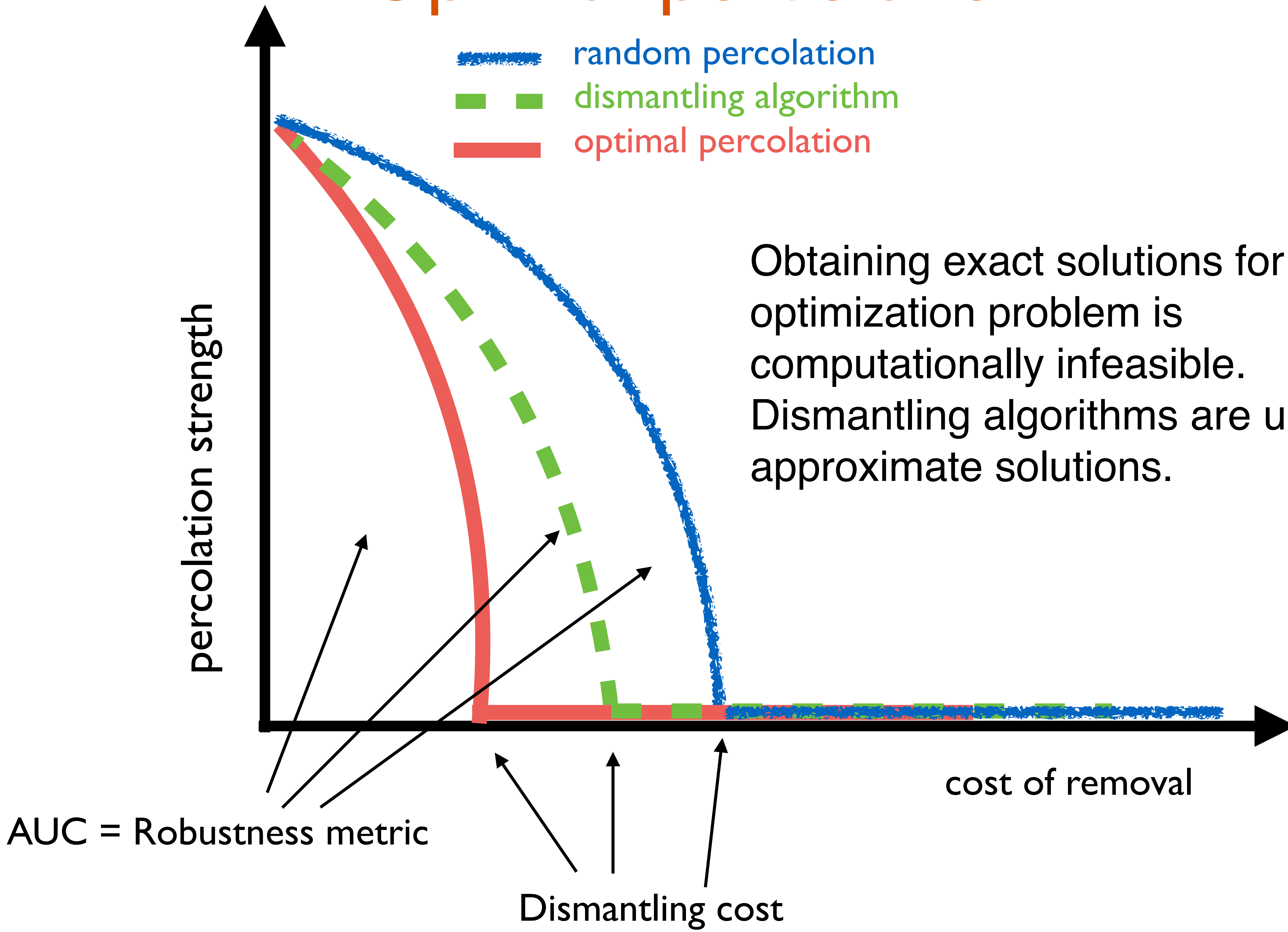
ideally, we should try all possible sets of any size, thus

$$\sum_{|\mathcal{S}|=0}^{N} \binom{N}{|\mathcal{S}|} = 2^N$$ structural sets

Solutions can be approximated via standard optimization techniques (i.e., simulated annealing and greedy optimization). Other approximations are based on network centrality metrics (e.g., degree, betweenness, closeness).

Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)
Xiao-Long Ren, et al. , PNAS 116, 6554–6559 (2019)

Optimal percolation



Obtaining exact solutions for the optimization problem is computationally infeasible. Dismantling algorithms are used to approximate solutions.

Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)
Xiao-Long Ren, et al. , PNAS 116, 6554–6559 (2019)

Optimal percolation

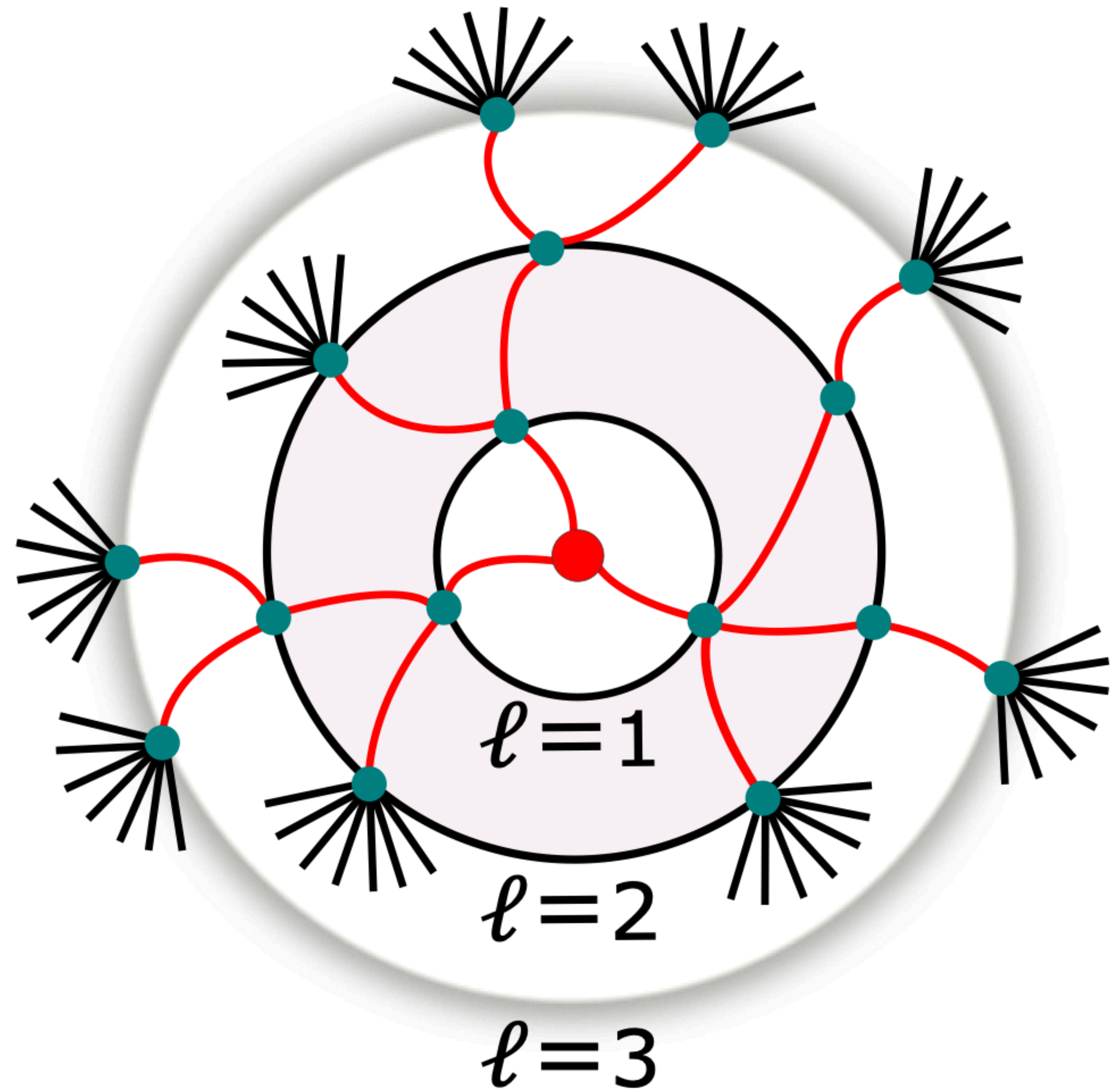
approximate method

Nodes are added to the structural set sequentially based on their centrality score

collective influence (CI)

adaptive, based on the remaining network where nodes already belonging to the structural set are removed.

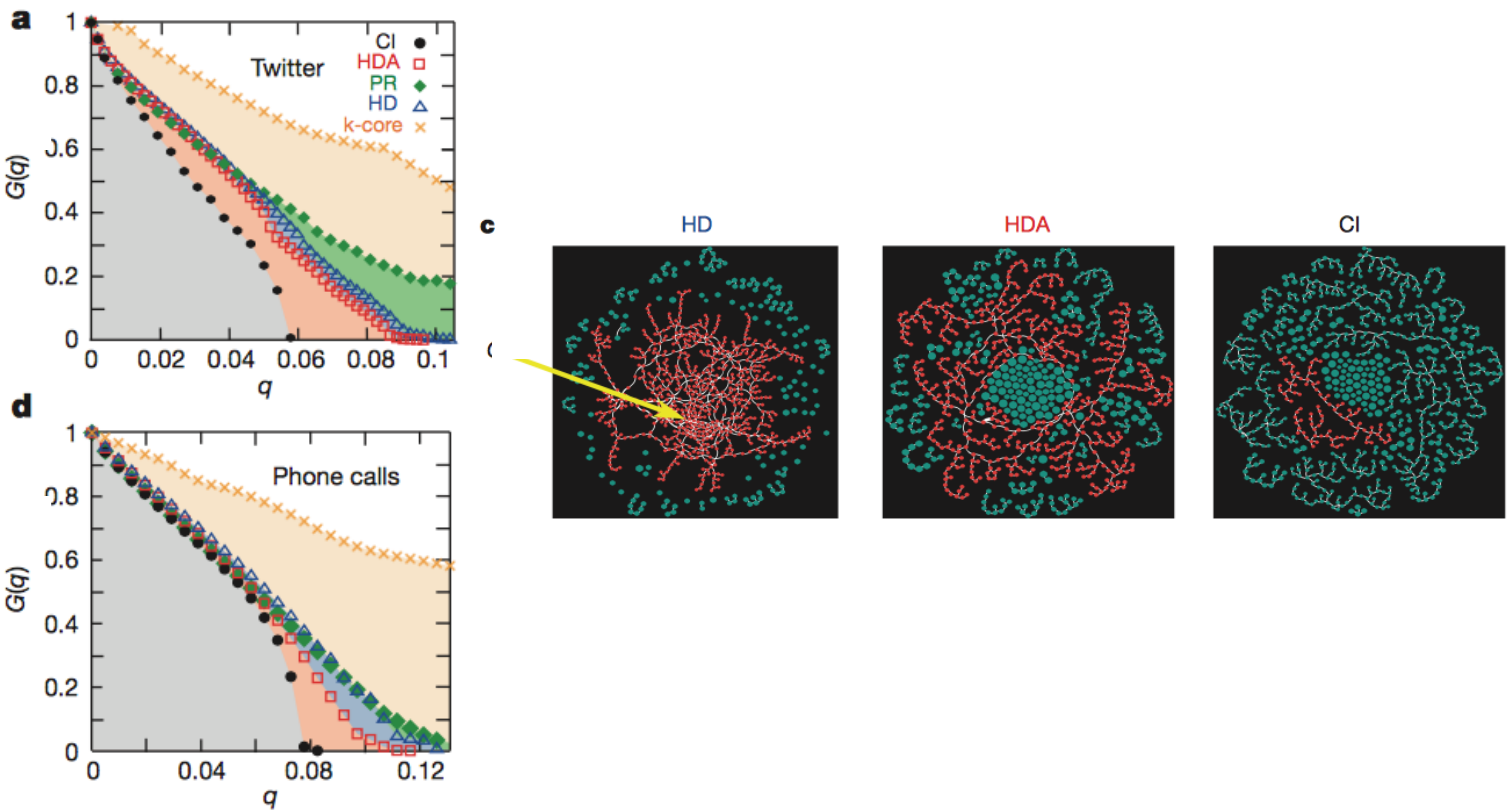
$$CI_\ell(i) = (k_i - 1) \sum_{j \in \partial B(i,\ell)} (k_j - 1)$$



Adaptive degree centrality is a special case of collective influence.

Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)

Optimal percolation



Flaviano Morone and Hernán A. Makse, Nature 524, 65 (2015)

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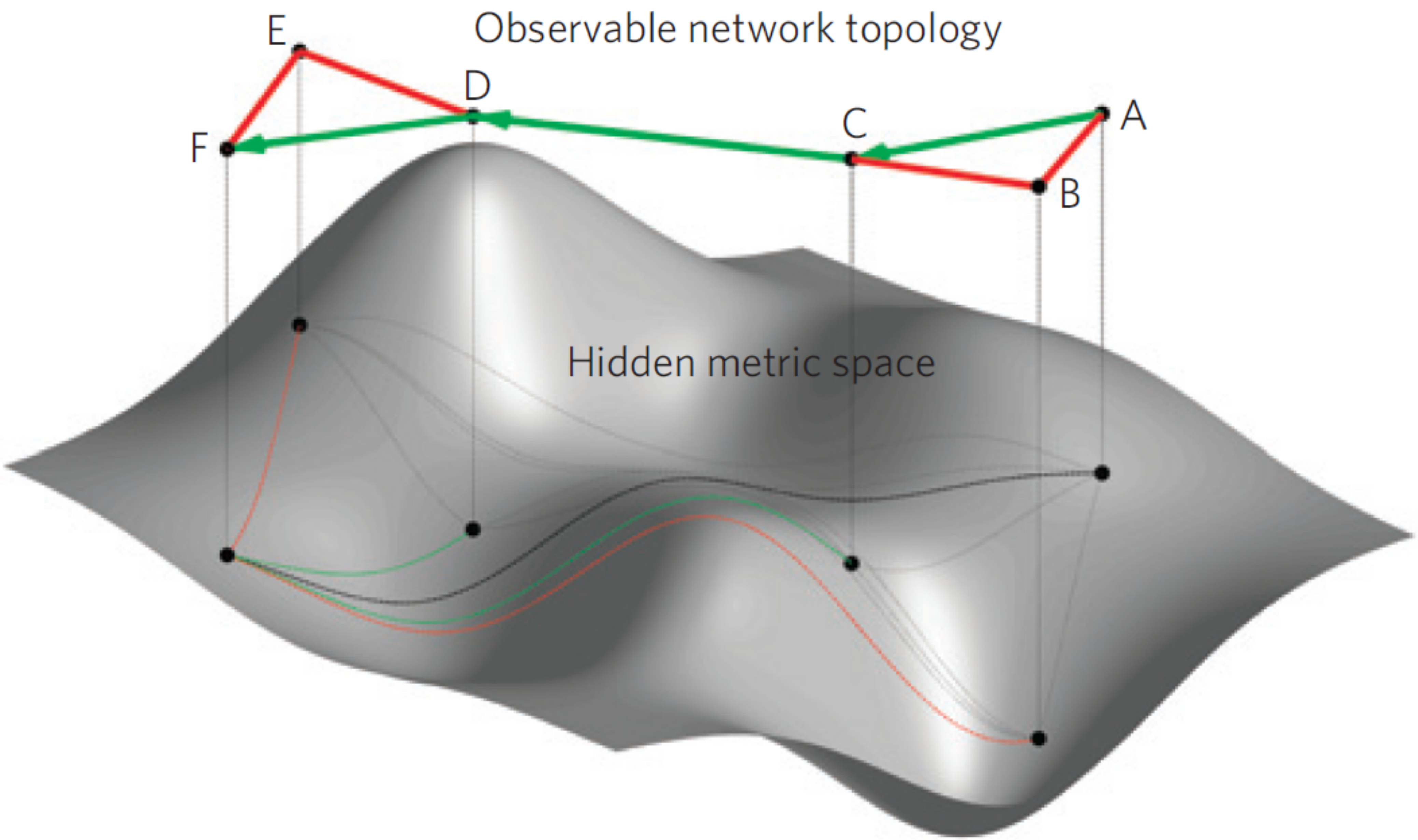
Geometric approach to network dismantling

The map encodes the structure of the network

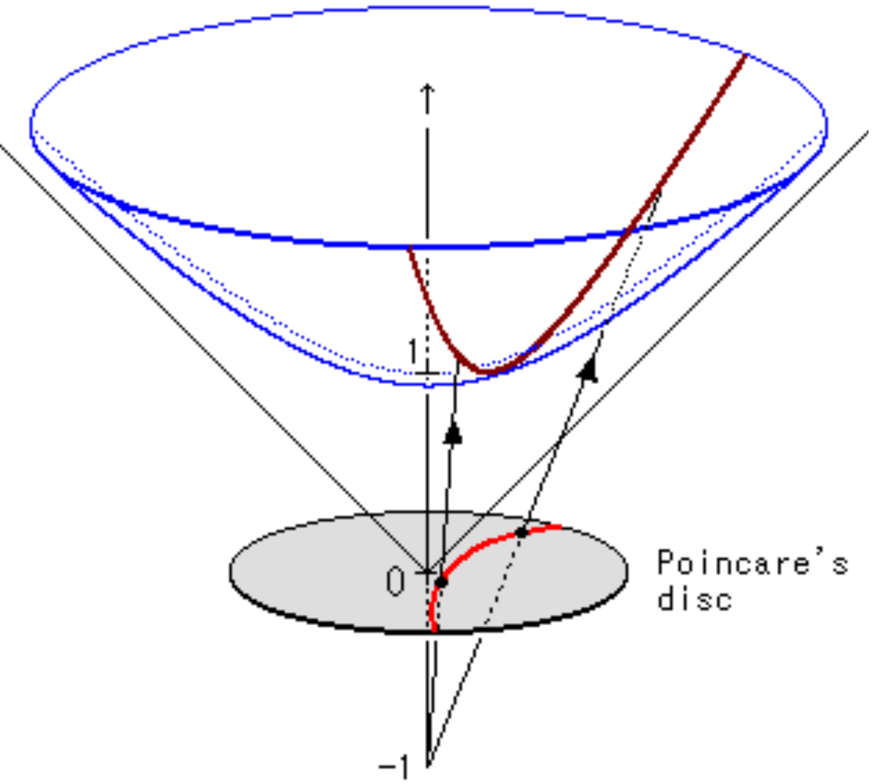


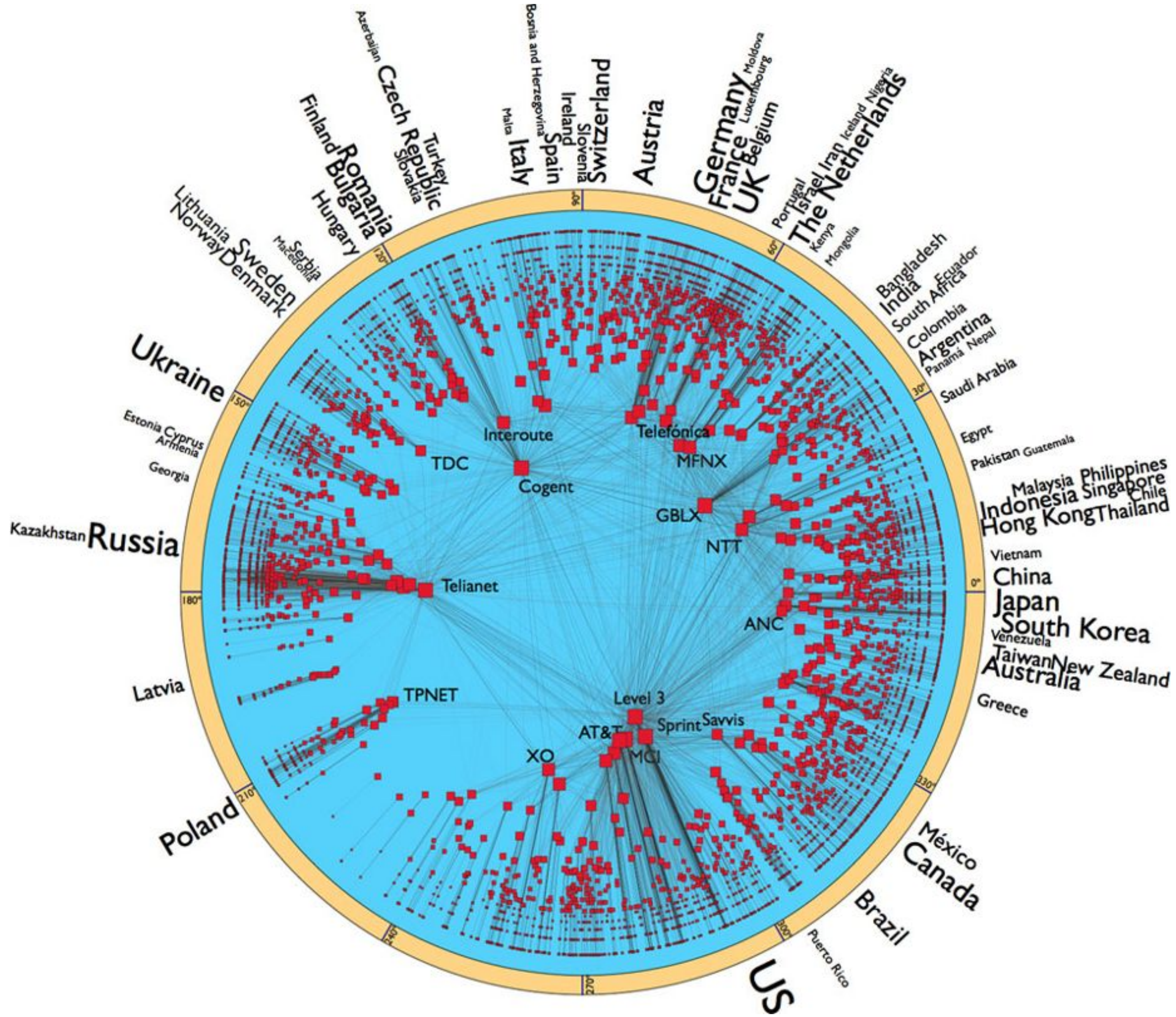
There are many potential ways to embed a network

M.A. Serrano, D. Krioukov, and M. Boguna, "Self-similarity of complex networks and hidden metric spaces," Physical Review Letters 100, 078701 (2008).

M. Boguna, D. Krioukov, and K.C. Claffy, "Navigability of complex networks," Nature Physics 5, 74–80 (2009).

Hyperbolic embedding





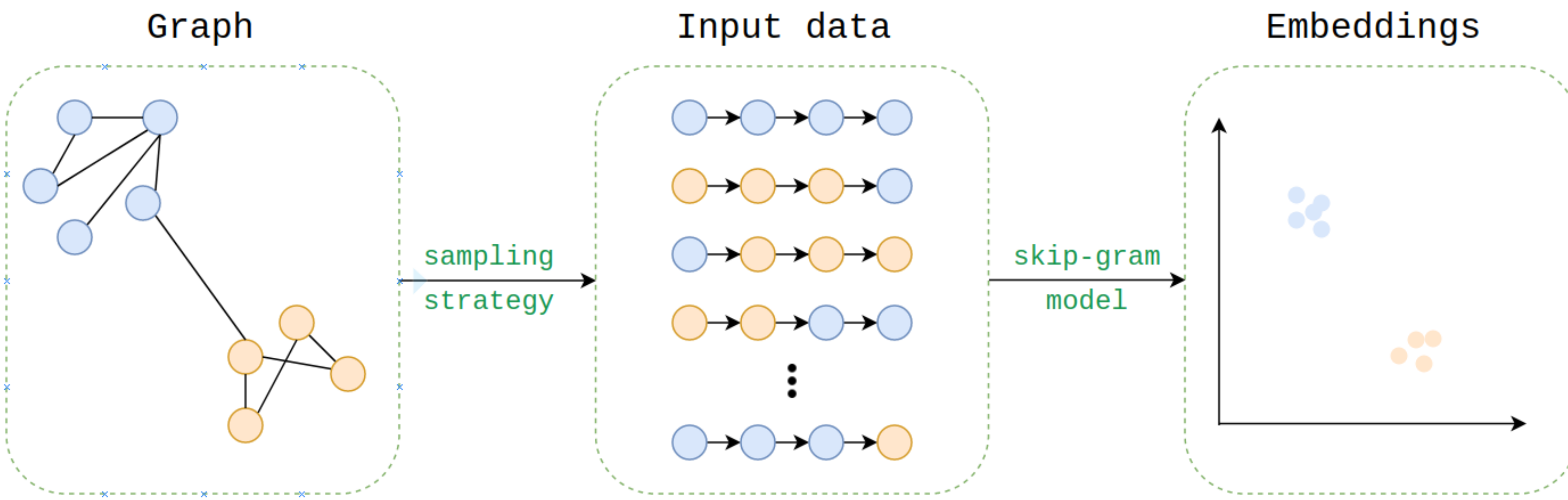
D. Krioukov et al. "Curvature and temperature of complex networks," Physical Review E 80, 035101 (2009).

D. Krioukov et al., "Hyperbolic geometry of complex networks," Physical Review E 82, 036106 (2010).

M. Boguna et al., "Sustaining the internet with hyperbolic mapping," Nature Communications 1, 62 (2010).

G. Bianconi and C. Rahmede, "Emergent hyperbolic network geometry," Scientific Reports 7 (2017).

Node2vec embedding



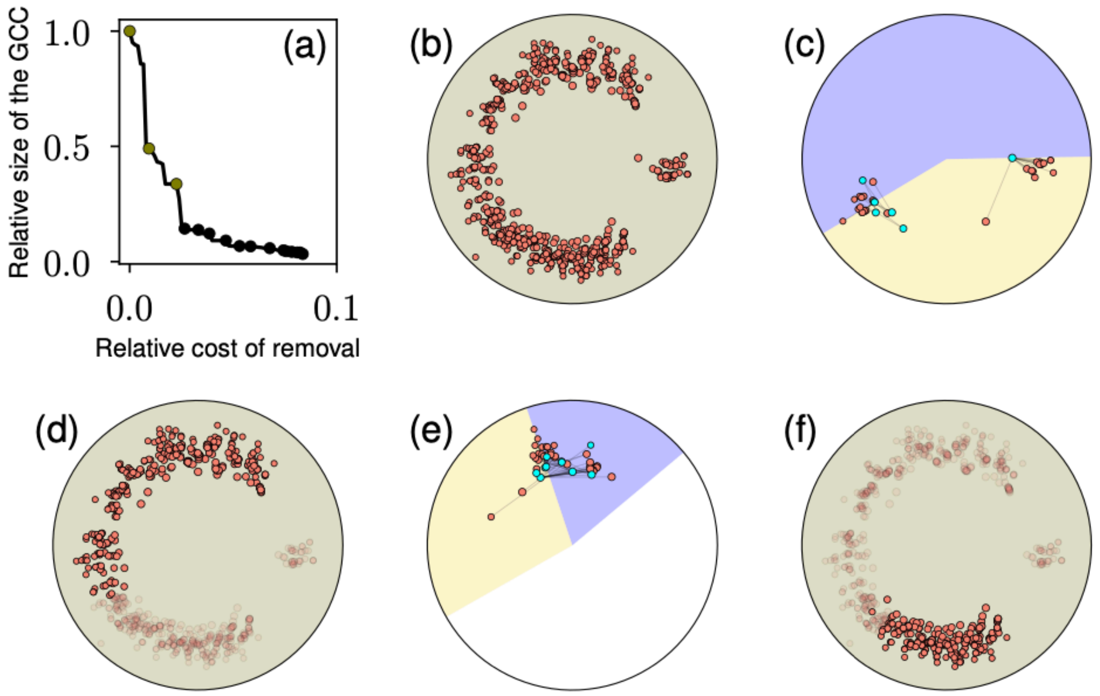
Node2vec builds on the **word2vec** algorithm by taking the following analogy: nodes in the network are considered as “words”; a sequence of nodes explored during a biased random walks is considered as a “sentence.”

T. Mikolov et al. "Efficient estimation of word representations in vector space." arXiv preprint arXiv:1301.3781 (2013).

A. Grover and J. Leskovec, "node2vec: Scalable feature learning for networks,” in Proceedings of the 22nd ACM SIGKDD international conference on Knowledge discovery and data mining (2016).

Image from https://towardsdatascience.com/node2vec-embeddings-for-graph-data-32a866340fef

Geometric approach to network dismantling



1. Embed the network in hyperbolic space.
2. Create two slices of similar size.
3. Remove the minimum number of nodes/edges that allow for the disconnection of the two slices.
4. Repeat from 2 until there are still edges/nodes in the graph.

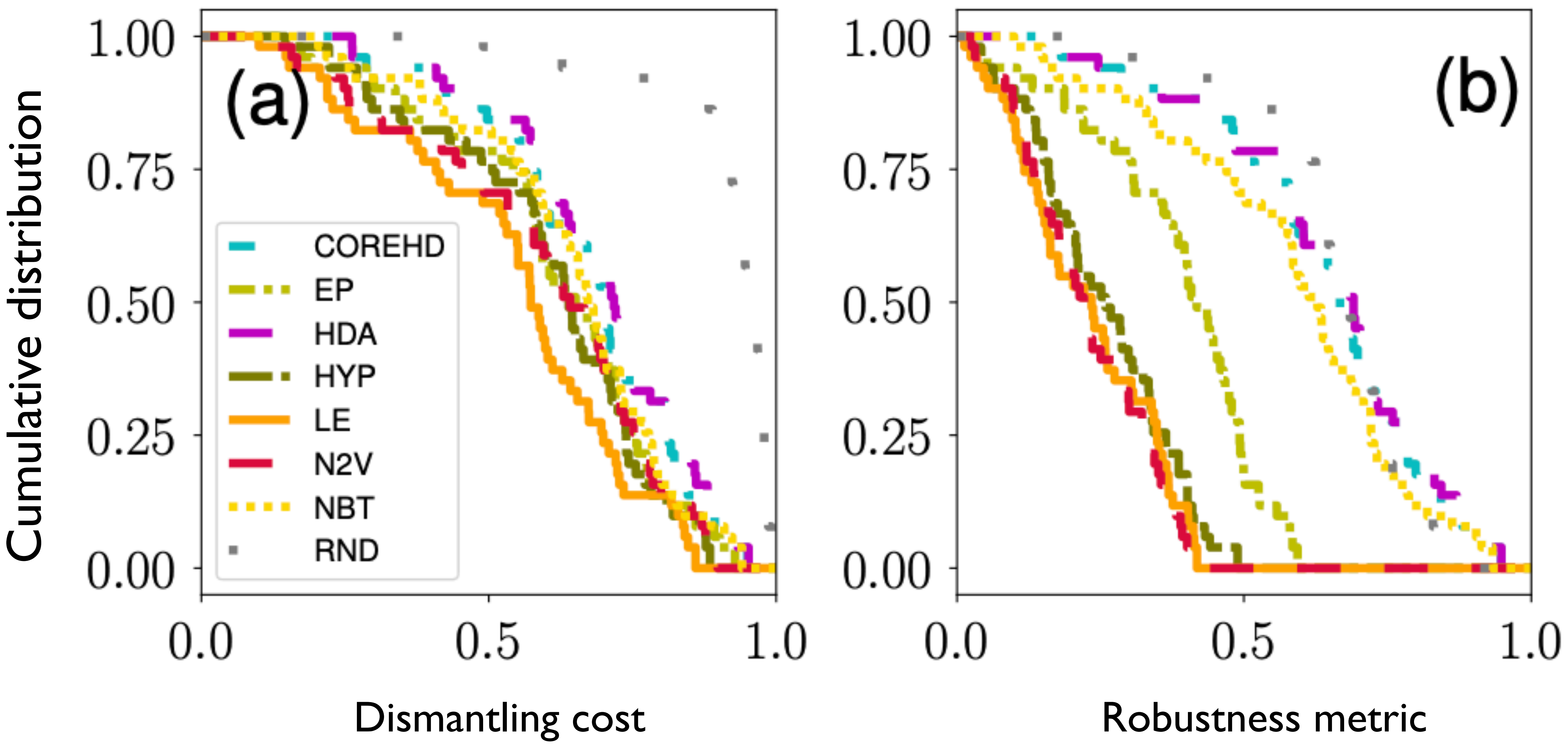
For node2vec embedding, the two slices are obtained via k-means clustering, with k=2.

<u>Hyperbolic</u>-embedding-aided dismantling requires a time that grows <u>quadratically</u> with the network size; the <u>node2vec</u>-embedding-based method scales <u>linearly</u> with the system size.

S. Osat, F. Papadopoulos, A.S. Teixeira, and F. Radicchi, Embedding-aided network dismantling, Phys. Rev. Research 5, 013076 (2023)

Geometric approach to network dismantling

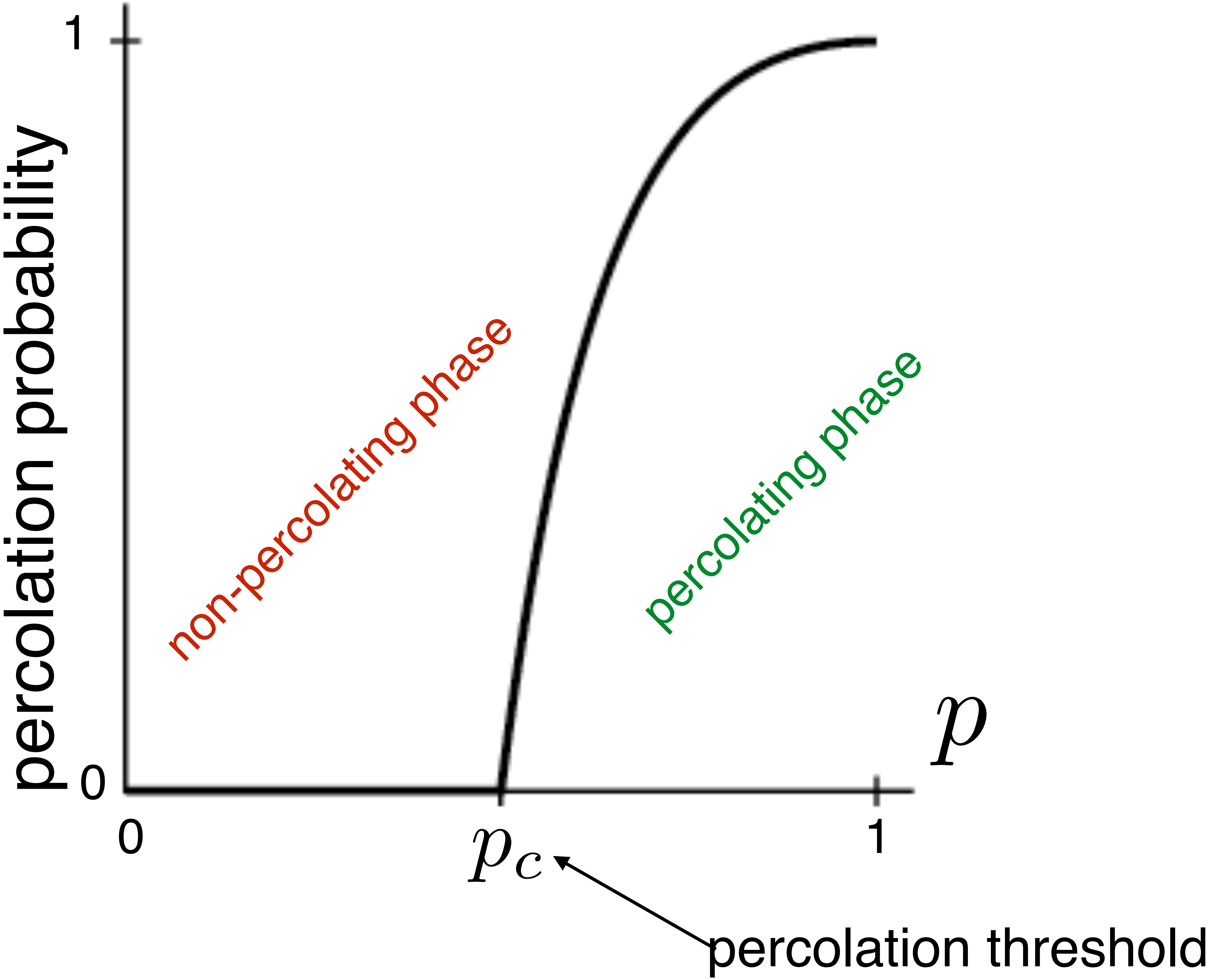
bond percolation (unit cost)
Results obtained over a corpus of 50 real-world networks



Geometric methods outperform centrality-based methods

S. Osat, F. Papadopoulos, A.S. Teixeira, and F. Radicchi, Embedding-aided network dismantling, Phys. Rev. Research 5, 013076 (2023)

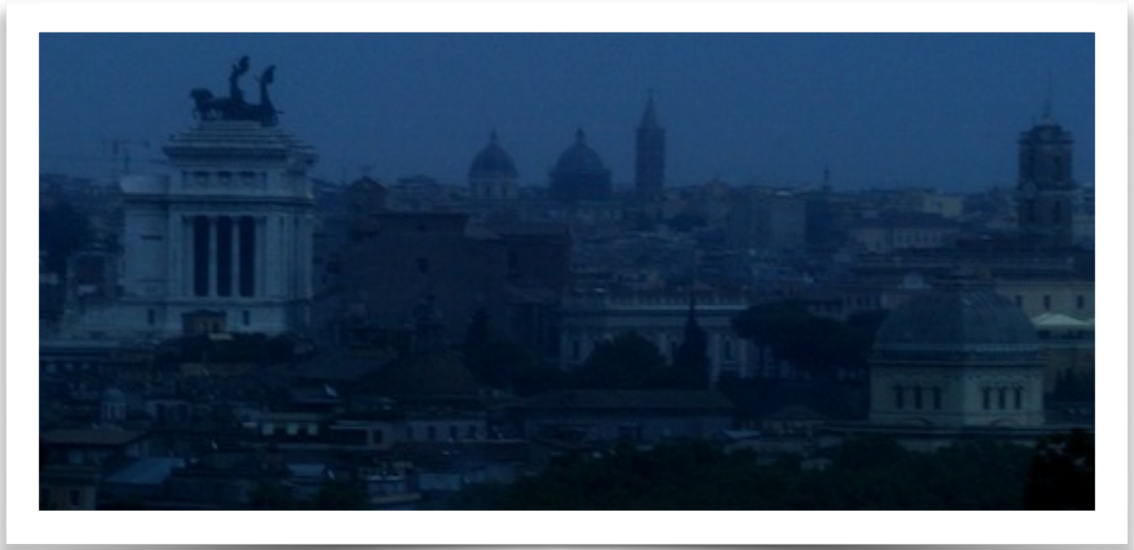
Is percolation a good model for catastrophes in critical infrastructures?

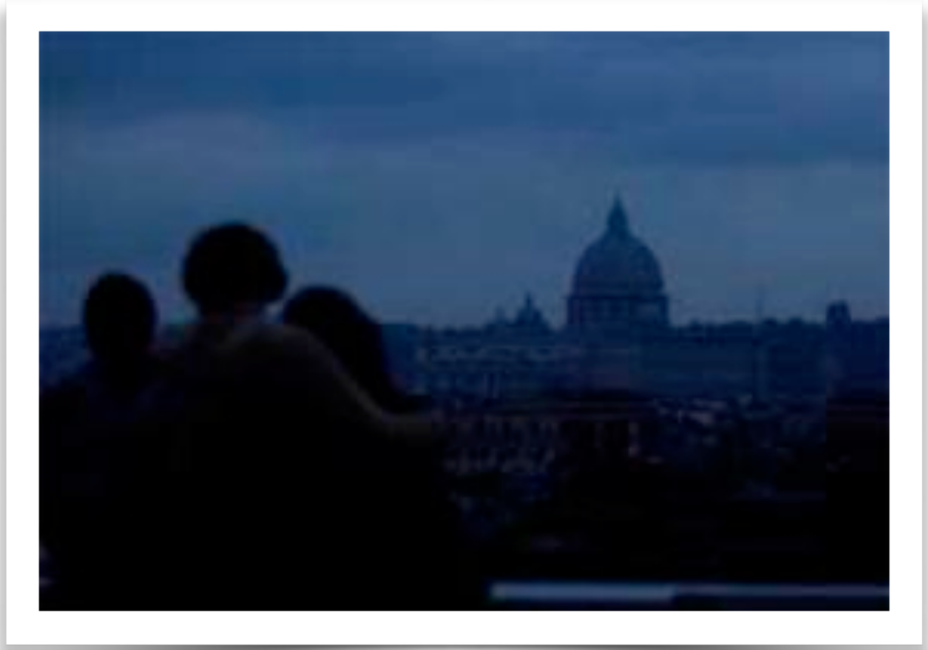


La "notte bianca"

Rome, September 27-28, 2003

Not really a “bright” night!





Buldyrev, Sergey V., et al. "Catastrophic cascade of failures in interdependent networks." *Nature* 464.7291 (2010): 1025-1028.

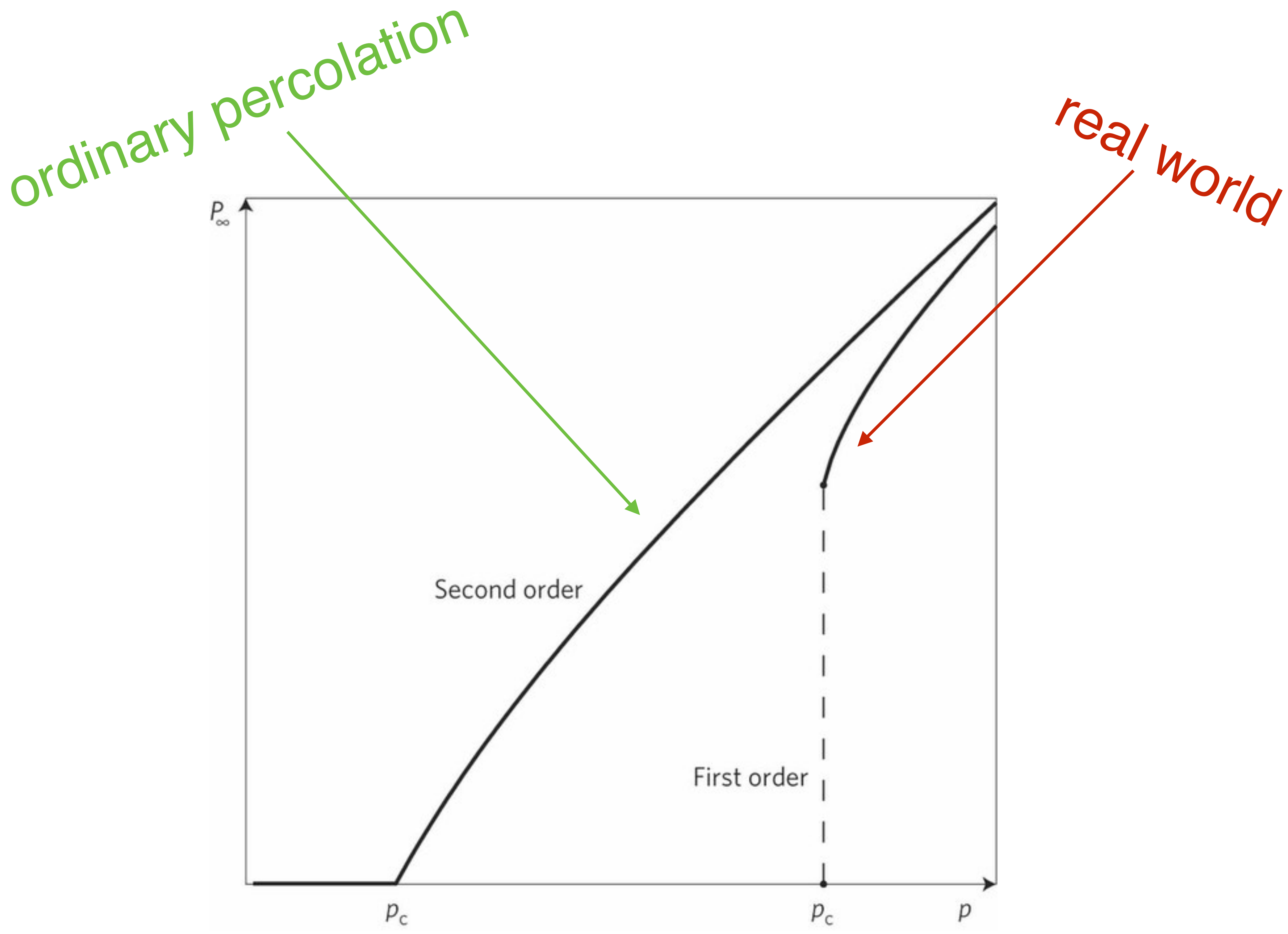
Catastrophic failures of critical infrastructures

Northeast 2003

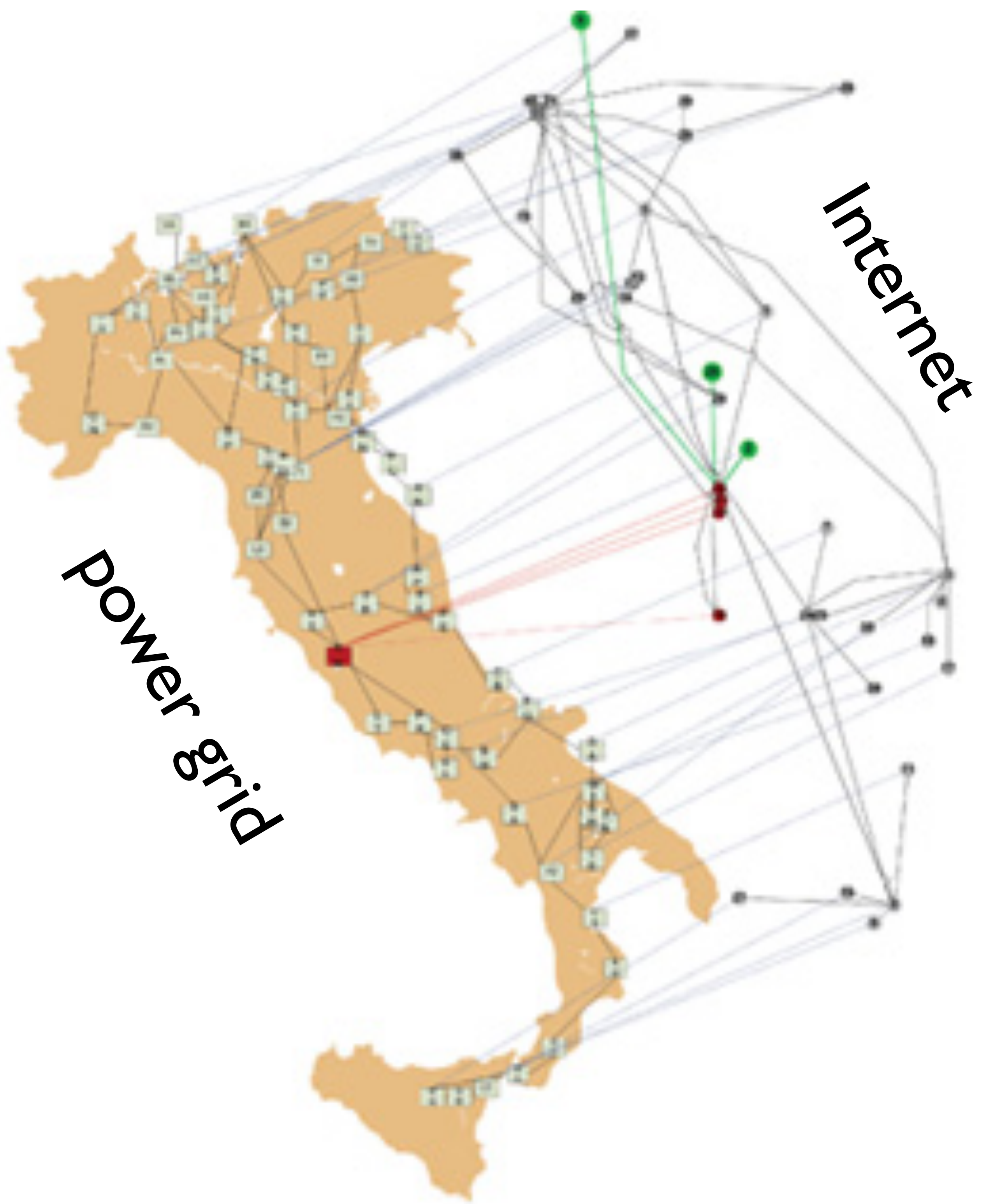
Catastrophic failures of critical infrastructures

Brazil 2009

Catastrophic percolation transition



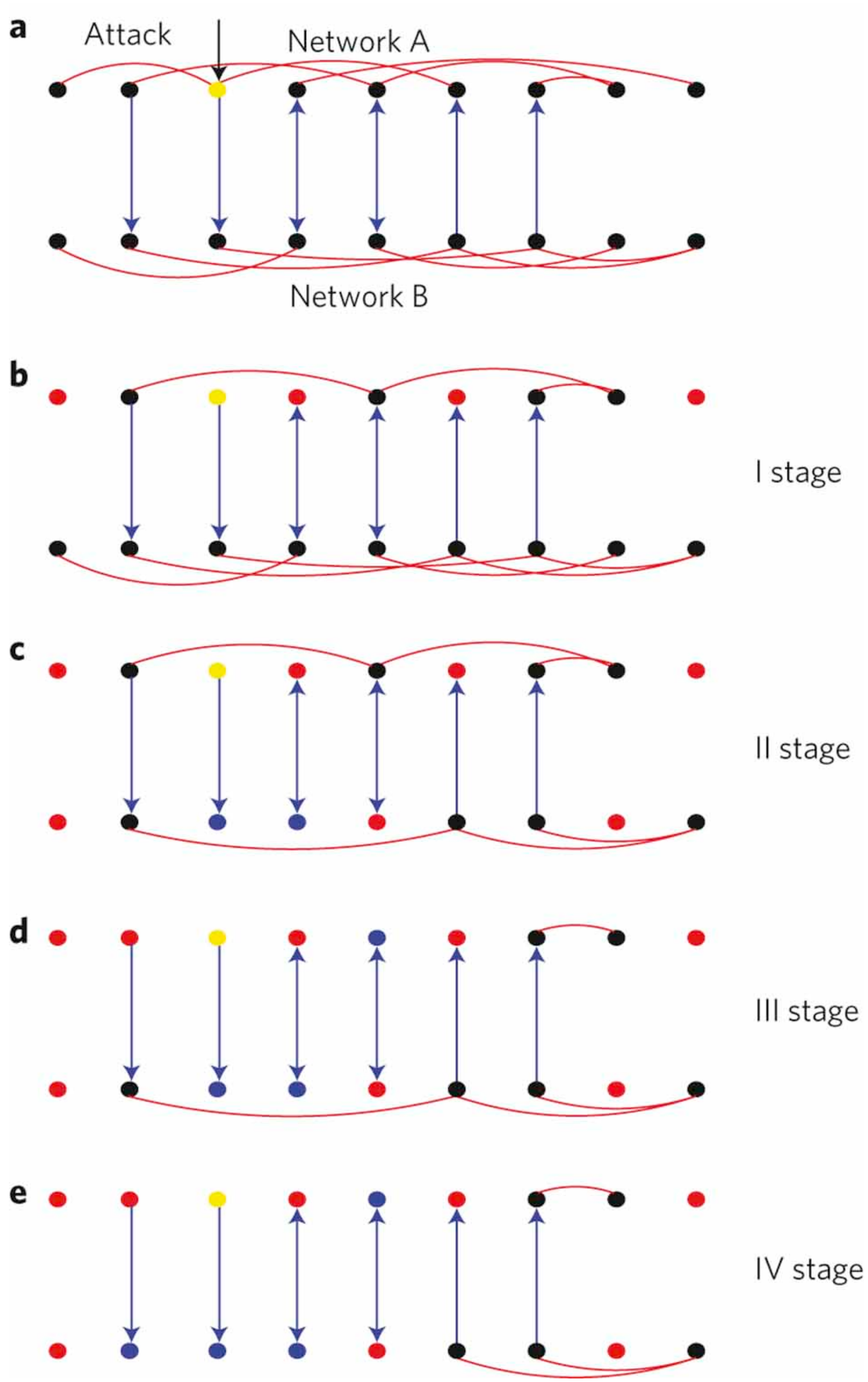
The Power grid and the Internet are “interdependent”

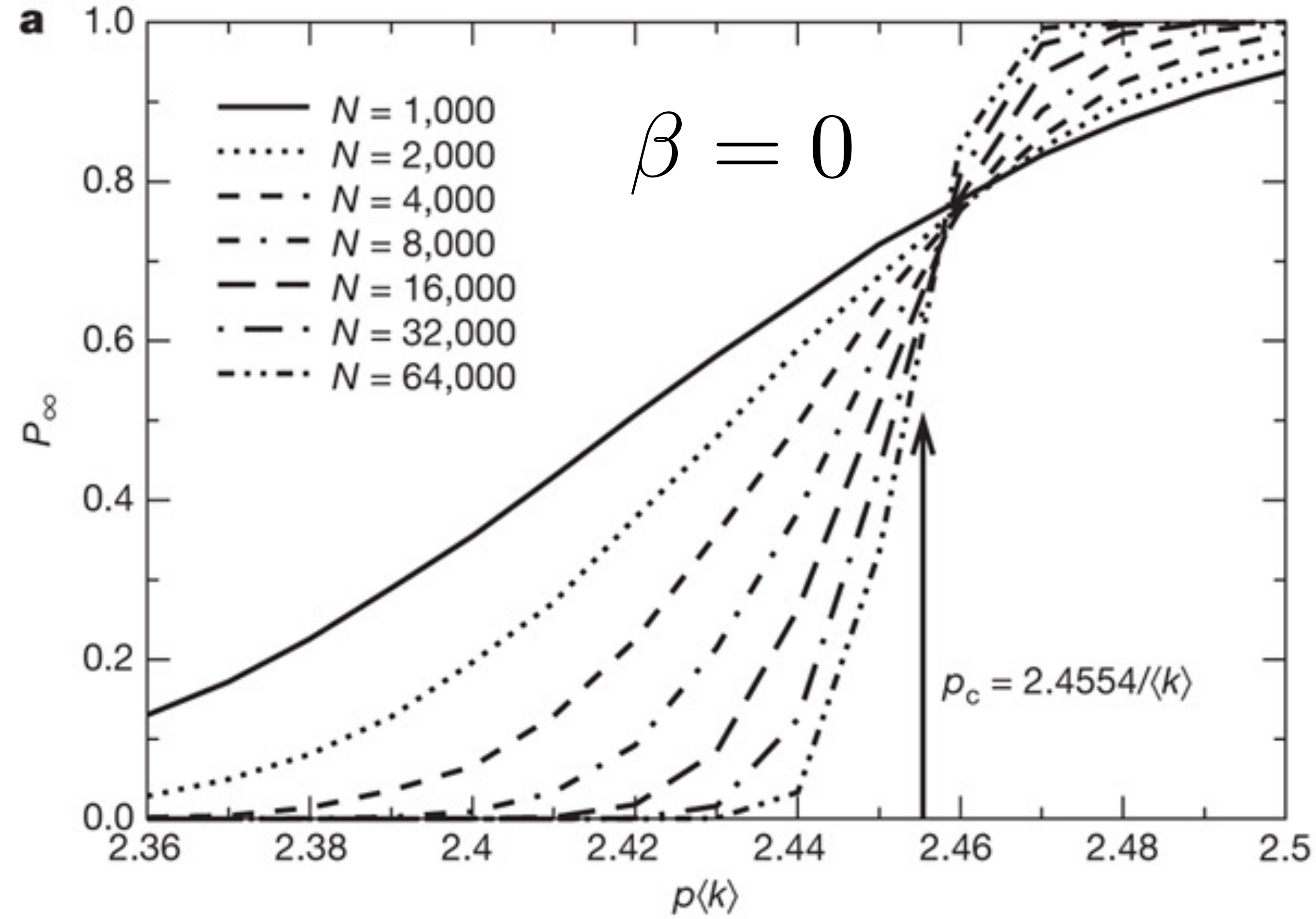


Buldyrev, Sergey V., et al. "Catastrophic cascade of failures in interdependent networks." *Nature* 464 (2010): 1025-1028.

Percolation transition in interdependent

first-order transition!





A node belongs to the mutually connected giant component (MCGC) if connected to at least one other node in the MCGC in each layer of the network

Buldyrev, Sergey V., et al. "Catastrophic cascade of failures in interdependent networks." *Nature* 464 (2010): 1025-1028.

jupyter percolation_on_complex_networks Last Checkpoint: 10 minutes ago (autosaved)

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Percolation on complex networks

This notebook serves at providing a brief introduction to the numerical study of percolation processes in complex networks. Given the large volume of work on this topic, the document illustrates only a small number of percolatio models.

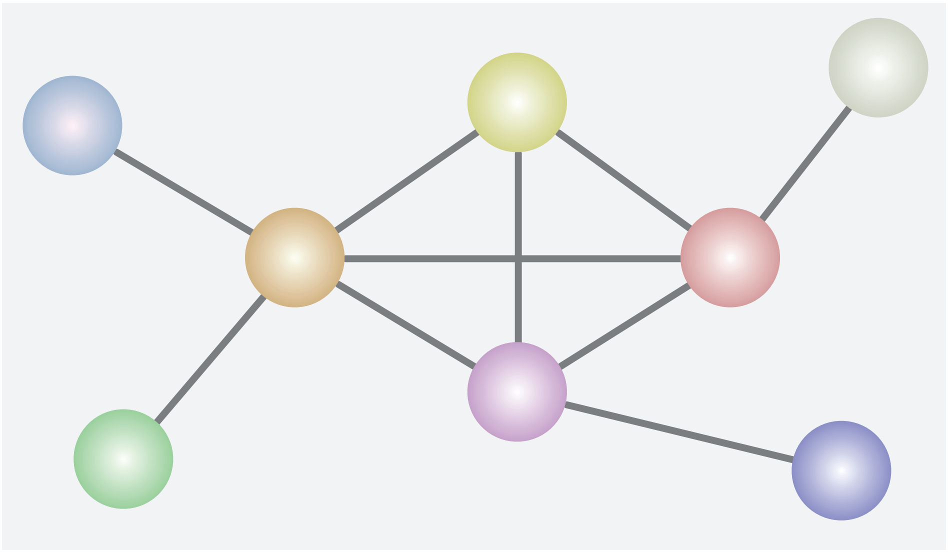
Outline

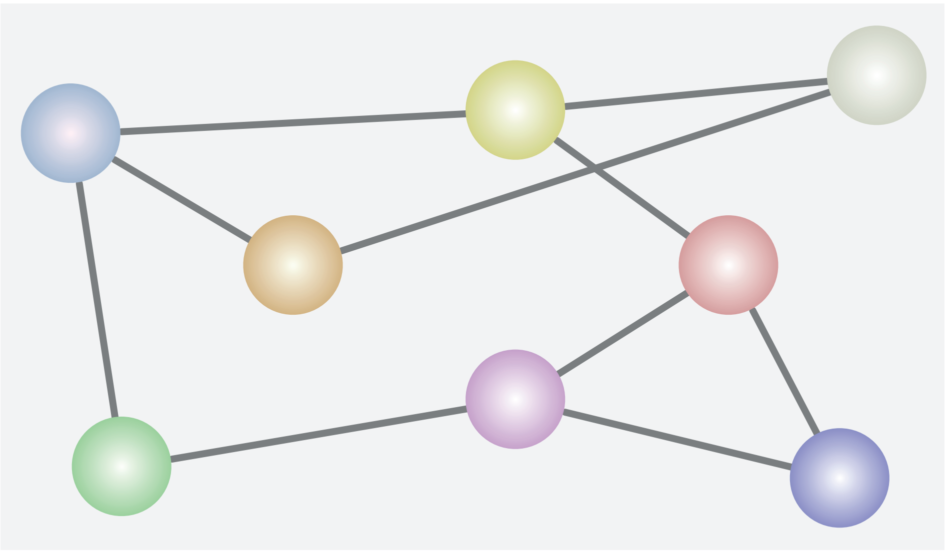
- Naive implementations of the ordinary bond and site percolation models
- Newman-Ziff algorithm for the ordinary bond and site percolation models
- Message-passing equations for ordinary percolation models on sparse networks
- Explosive percolation
- Optimal percolation
- Ordinary site percolation on multiplex networks
- Message-passing equations for ordinary site percolation on sparse multiplex networks

Libraries used in this notebook

In [1]:
```
import networkx as nx
import random
import numpy as np
```

Site percolation in interdependent networks





$$s_i = p\left[R_{\mathcal{AB}_i} + \left(1 - R_{\mathcal{AB}_i}\right) R_{\mathcal{A}-\mathcal{B}_i} \, R_{\mathcal{B}-\mathcal{A}_i}\right]$$

$$r_{i\to j} = p\left[R_{\mathcal{AB}_j\setminus\{i\}} + \left(1 - R_{\mathcal{AB}_j\setminus\{i\}}\right) R_{\mathcal{A}-\mathcal{B}_j\setminus\{i\}} \, R_{\mathcal{B}-\mathcal{A}_j\setminus\{i\}}\right]$$

---

where

$$R_{\mathcal{X}_i} = 1 - \prod_{j\in\mathcal{X}} \left(1 - r_{i\to j}\right)$$

$\mathcal{AB}_i = \mathcal{N}_i^A \cap \mathcal{N}_i^B$ — neigh. in both layers

$\mathcal{A} - \mathcal{B}_i = \mathcal{N}_i^A \setminus \mathcal{AB}_i$ — neigh. only in layer A

$\mathcal{B} - \mathcal{A}_i = \mathcal{N}_i^B \setminus \mathcal{AB}_i$ — neigh. only in layer B

F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

jupyter percolation_on_complex_networks Last Checkpoint: 10 minutes ago (autosaved)

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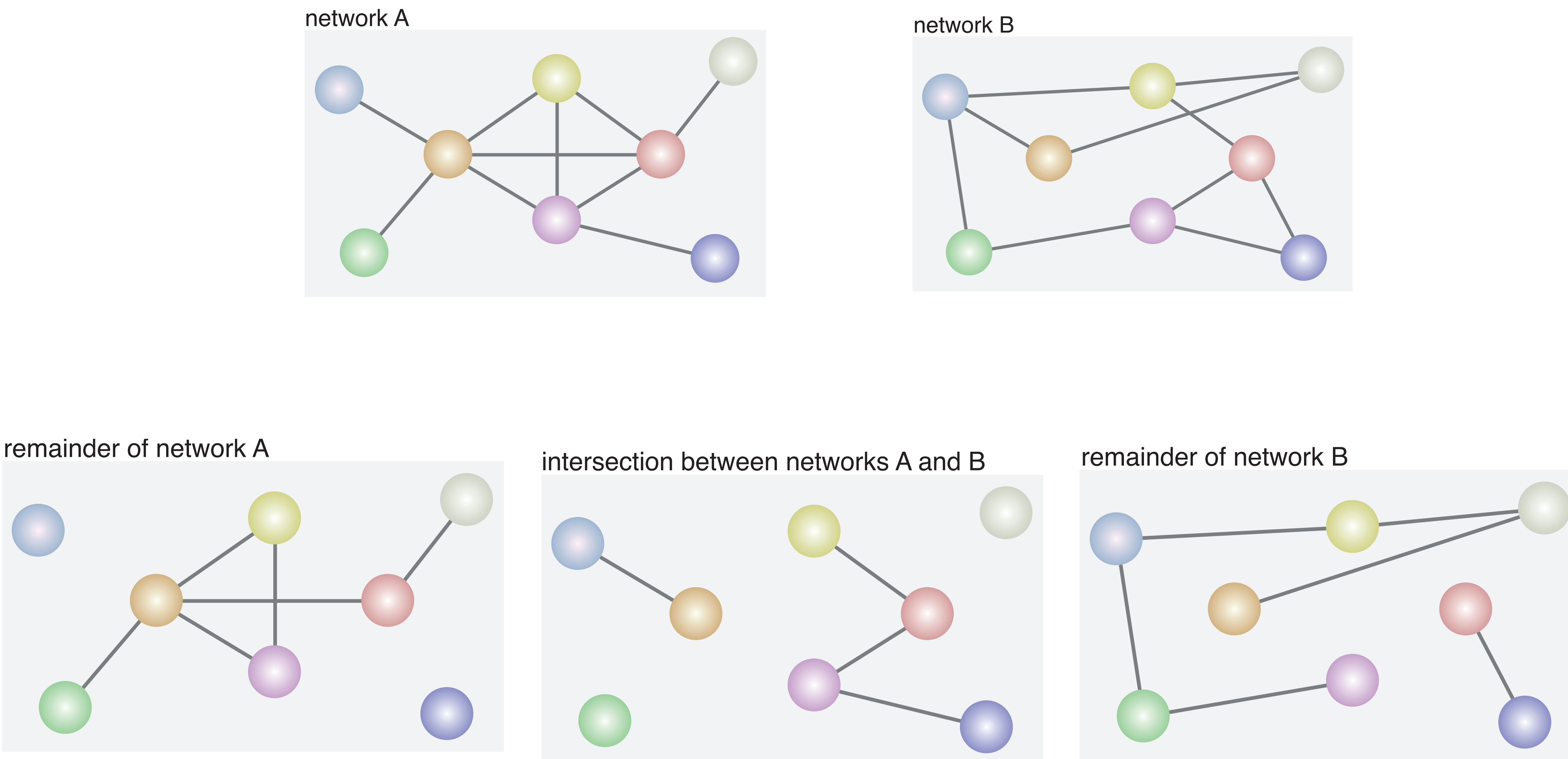
Libraries used in this notebook

In [1]:
```
import networkx as nx
import random
import numpy as np
```

Decomposition of the interdependent network



Cellai,D.,Lopez,E.,Zhou,J.,Gleeson,J.P.&Bianconi,G.Percolation in multiplex networks with overlap. *Phys. Rev. E* 88, 052811 (2013)

Min, B., Lee, S., Lee, K.-M. & Goh, K.-I. Link overlap, viability, and mutual percolation in multiplex networks. *Chaos, Solitons & Fractals* (2015)

What do the equations tell us?

$$r_{i\to j} = p\,[R_{\mathcal{A}\mathcal{B}_j\setminus\{i\}} + (1 - R_{\mathcal{A}\mathcal{B}_j\setminus\{i\}})\, R_{\mathcal{A}-\mathcal{B}_j\setminus\{i\}}\, R_{\mathcal{B}-\mathcal{A}_j\setminus\{i\}}]$$

$$R_{\mathcal{A}\mathcal{B}_j\setminus\{i\}} = 1 - \exp\,[\sum_{k\to\ell} M^{(AB)}_{i\to j,k\to\ell}\, w^{(AB)}_{k\to\ell}]$$

with

$$w^{(AB)}_{i\to j} = \ln(1 - r^{(AB)}_{i\to j})$$

$M^{(AB)}$ = non-backtracking matrix of the intersection graph

F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

What do the equations tell us?

site percolation

$$s_i = p\left[R_{\mathcal{A}\mathcal{B}_i} + \left(1 - R_{\mathcal{A}\mathcal{B}_i}\right) R_{\mathcal{A}-\mathcal{B}_i}\, R_{\mathcal{B}-\mathcal{A}_i}\right]$$

$$r_{i\to j} = p\left[R_{\mathcal{A}\mathcal{B}_j\setminus\{i\}} + \left(1 - R_{\mathcal{A}\mathcal{B}_j\setminus\{i\}}\right) R_{\mathcal{A}-\mathcal{B}_j\setminus\{i\}}\, R_{\mathcal{B}-\mathcal{A}_j\setminus\{i\}}\right]$$

with $R_{\mathcal{X}_i} = 1 - \prod_{j\in\mathcal{X}} \left(1 - r_{i\to j}\right)$

---

bond percolation

$$v_i = T_{\mathcal{A}\mathcal{B}_i} + \left(1 - T_{\mathcal{A}\mathcal{B}_i}\right) T_{\mathcal{A}-\mathcal{B}_i}\, T_{\mathcal{B}-\mathcal{A}_i}$$

$$t_{i\to j} = T_{\mathcal{A}\mathcal{B}_j\setminus\{i\}} + \left(1 - T_{\mathcal{A}\mathcal{B}_j\setminus\{i\}}\right) T_{\mathcal{A}-\mathcal{B}_j\setminus\{i\}}\, T_{\mathcal{B}-\mathcal{A}_j\setminus\{i\}}$$

with $T_{\mathcal{X}_i} = 1 - \prod_{j\in\mathcal{X}} \left(1 - p\, t_{i\to j}\right)$

$$P_{\infty}^{(site)} = p P_{\infty}^{(bond)}$$

F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

What do the equations tell us?

Truncated Taylor expansion

intersection graph

$$
\begin{aligned}
r_{i\to j} = \quad & p \sum_k A_{jk} B_{jk} (1-\delta_{ki}) r_{j\to k} \\
& + \frac{p}{2} \sum_{k,v} A_{jk} B_{jk} (1-\delta_{ki}) A_{jv} B_{jv} (1-\delta_{vi}) r_{j\to k} r_{j\to v} (1-\delta_{kv}) \\
& + \frac{p}{2} \left[\sum_k A_{jk}(1-B_{jk})(1-\delta_{ki}) r_{j\to k}\right] \left[\sum_k B_{jk}(1-A_{jk})(1-\delta_{ki}) r_{j\to k}\right] \\
& + o(r_{i\to j}^3)
\end{aligned}
$$

only layer A

only layer B

F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

What do the equations tell us?

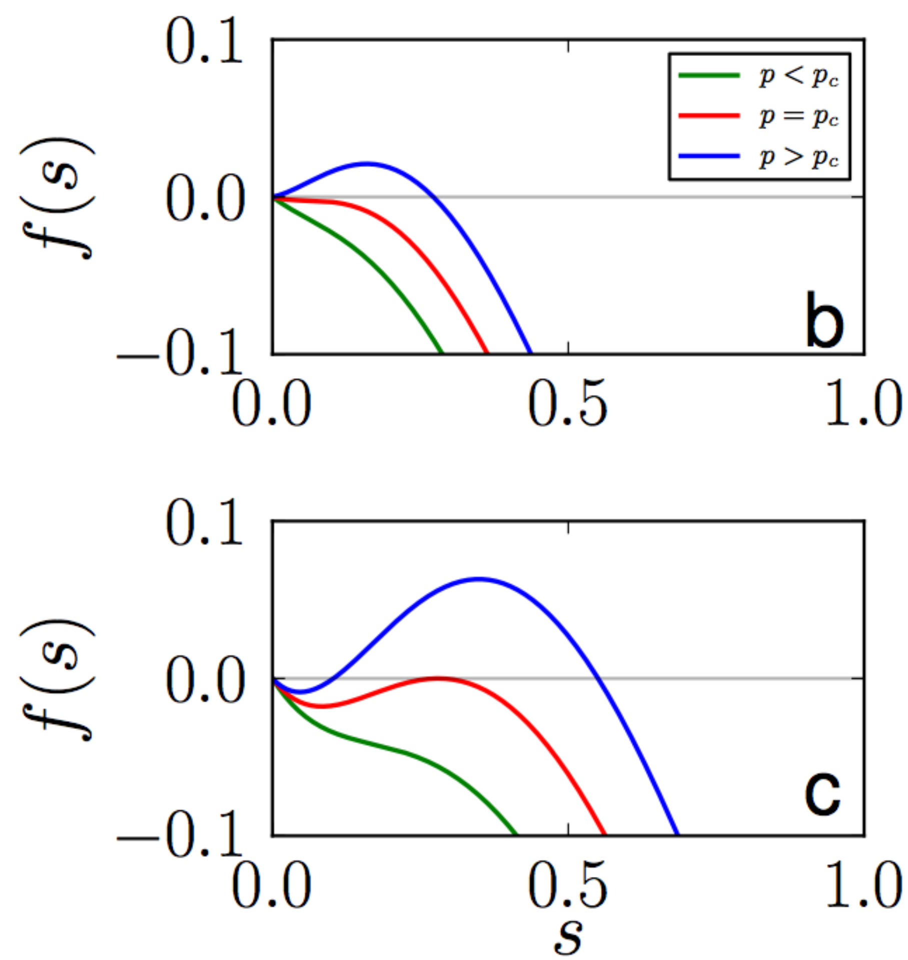
Coupled regular graphs

k = valency of the intersection graph

t = valency of the remainders

$$s = p\left\{1 - (1-r)^k + (1-r)^k[1-(1-r)^t]^2\right\}$$

$$r = p\left\{1 - (1-r)^{k-1} + (1-r)^{k-1}[1-(1-r)^{t-1}]^2\right\}$$

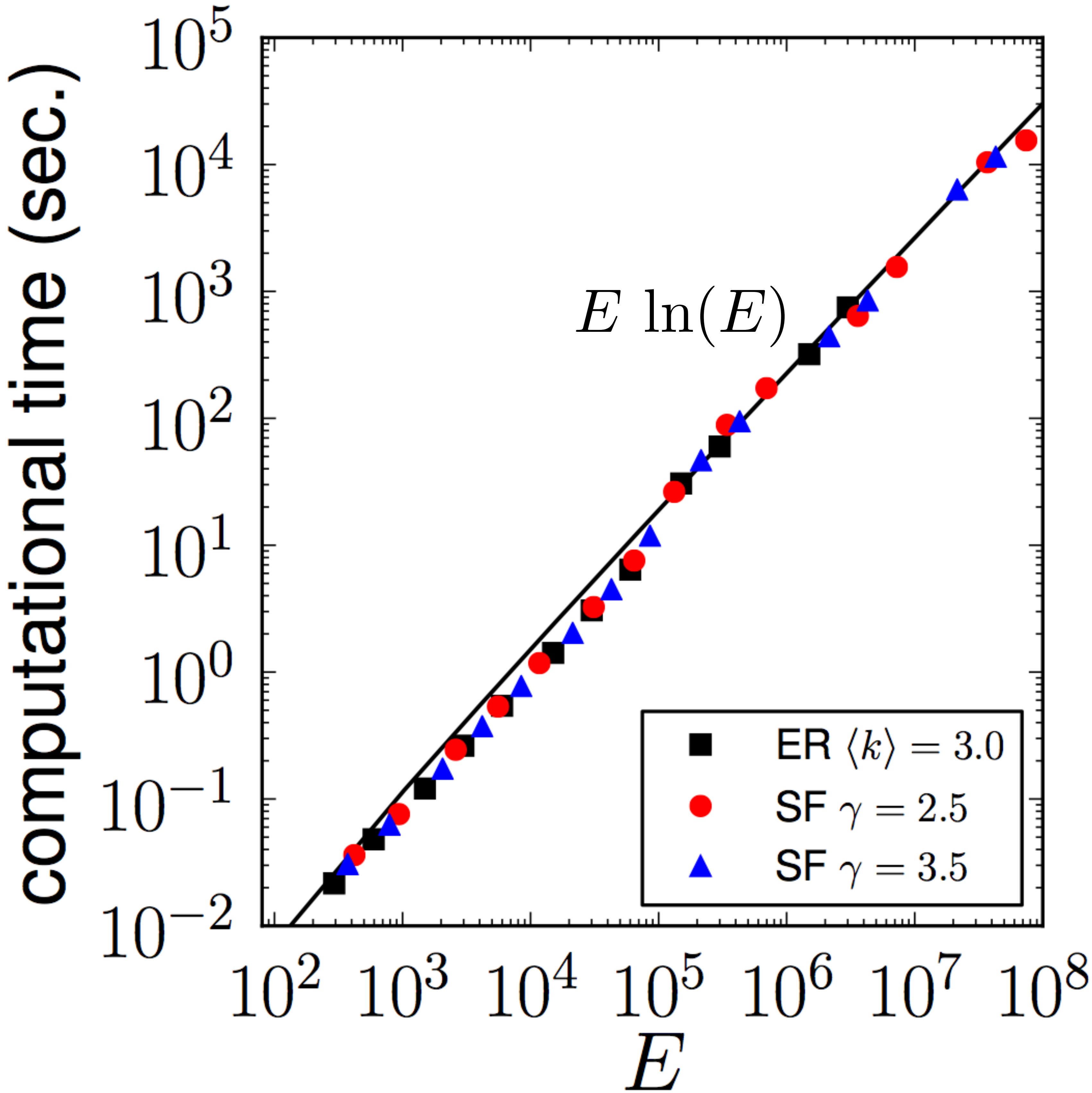


for k = t = 2

$$r = \frac{1 \pm \sqrt{5 - 4/p}}{2} \longrightarrow \begin{array}{l} p_c = 4/5 \\ P_\infty(p_c) = 57/80 \end{array}$$

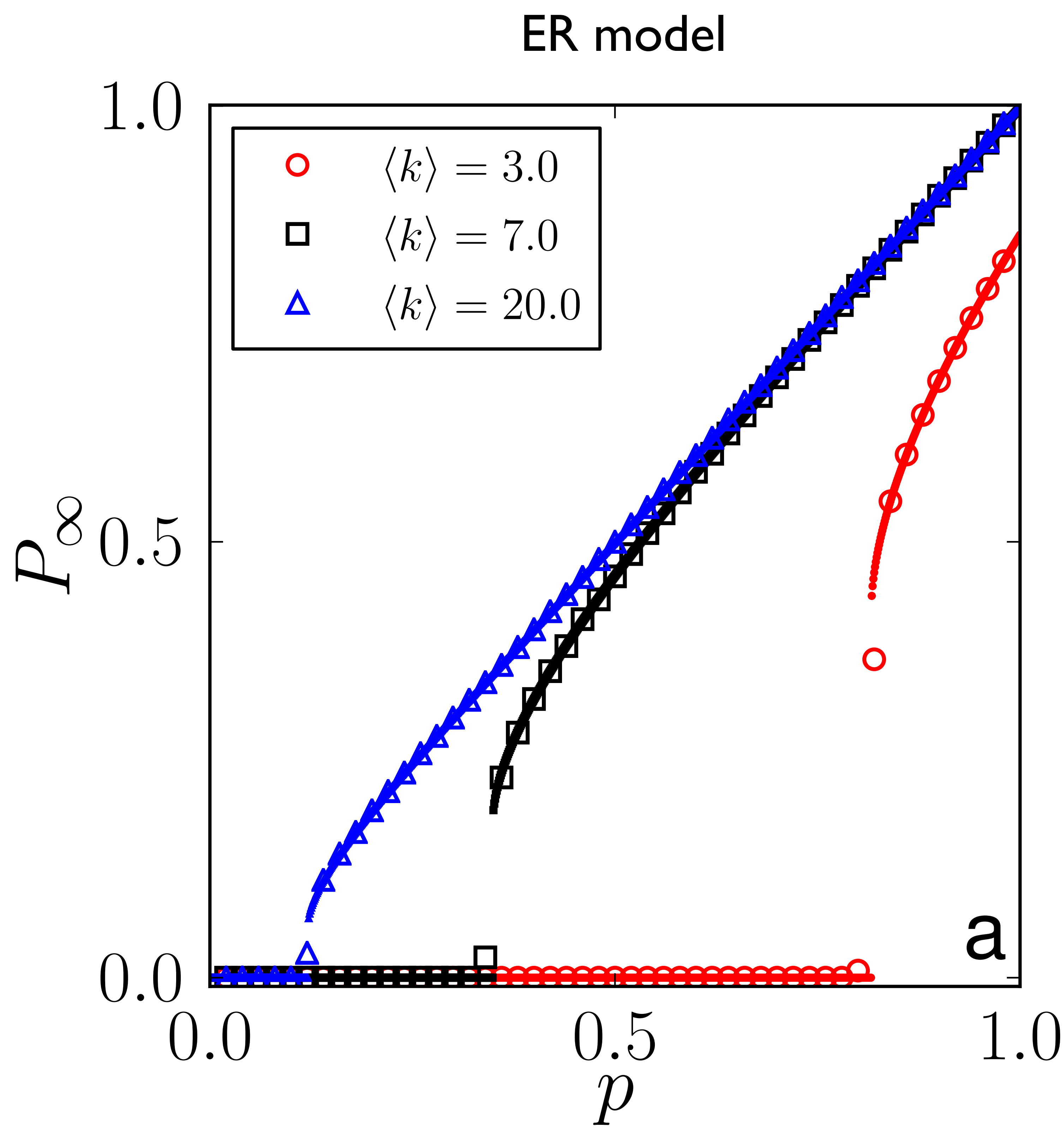
F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

Equations can be solved efficiently!



F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

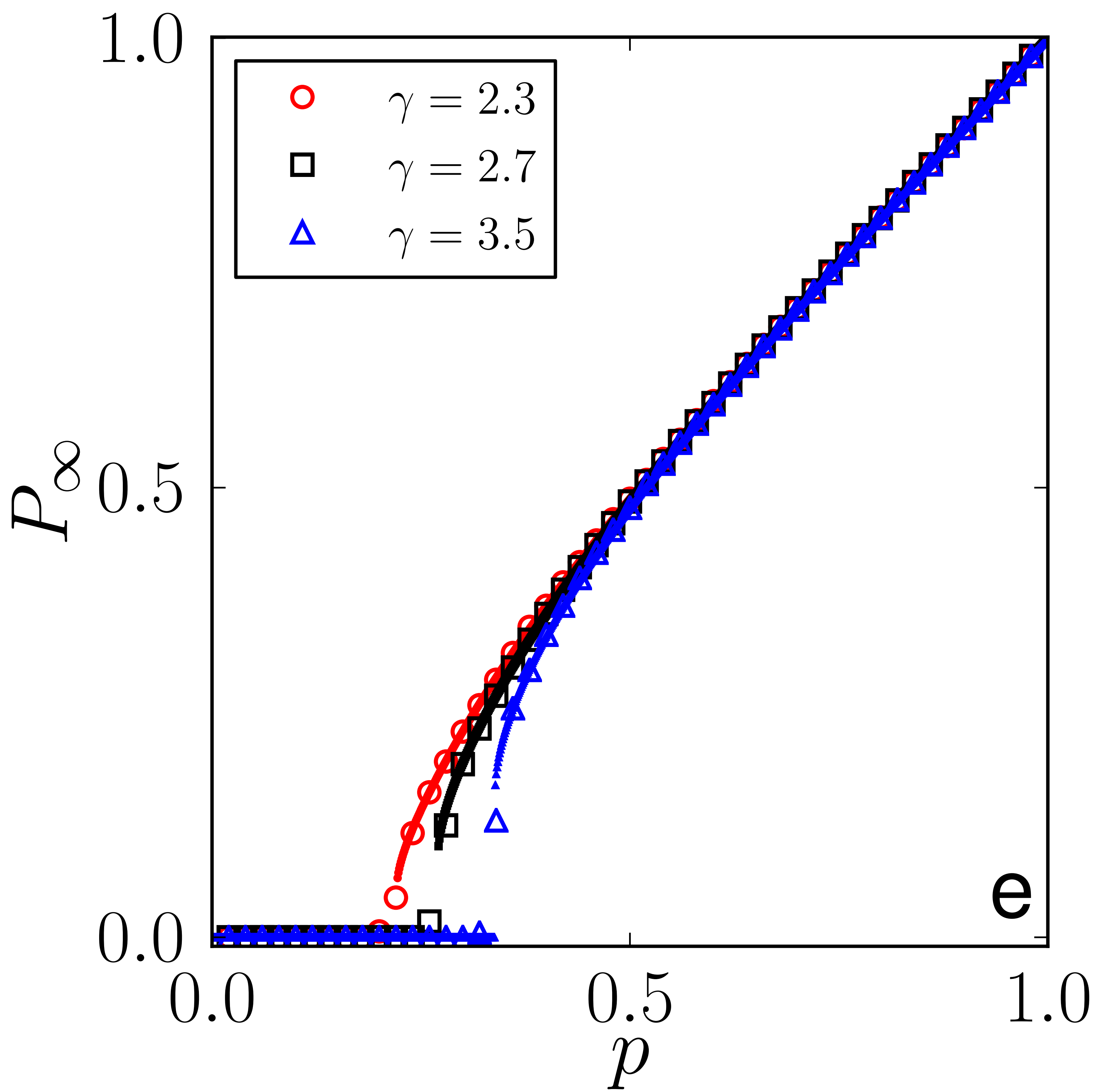
Results on finite-size network models



F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

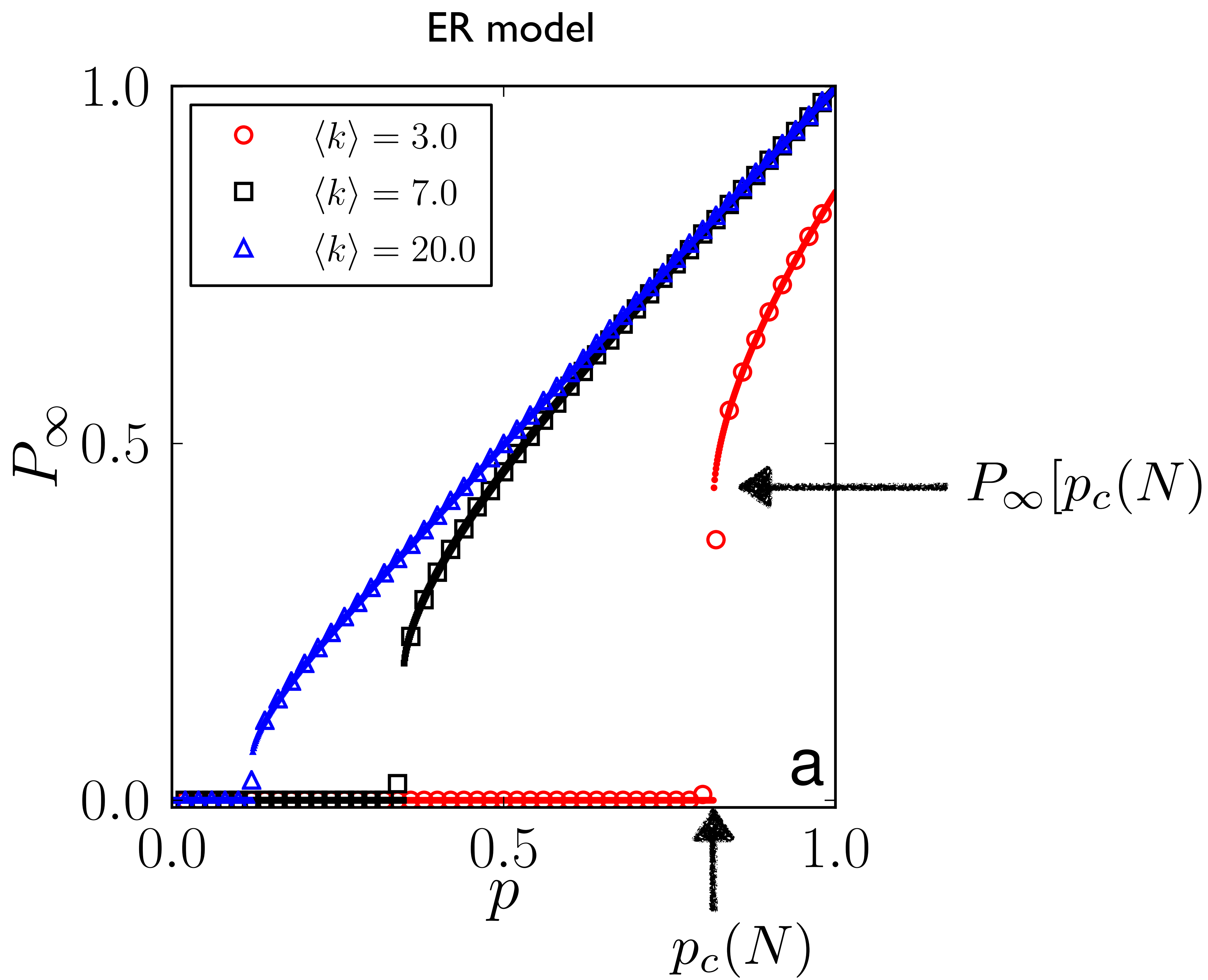
Results on finite-size network models

Configuration model, SF graphs



F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

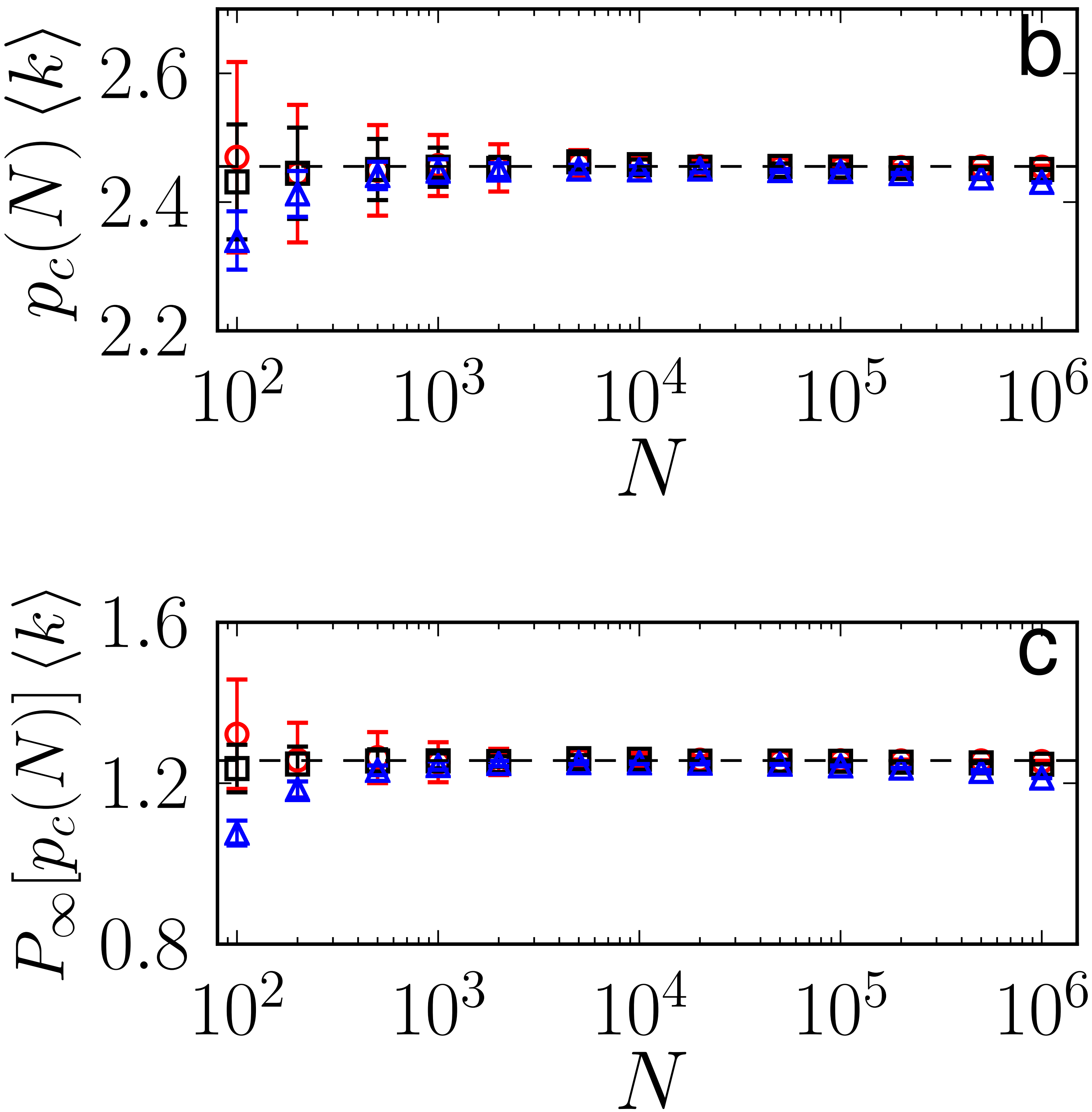
Finite-size scaling analysis



F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

Finite size scaling analysis

ER model



$$p_c\langle k\rangle = 2.4554$$

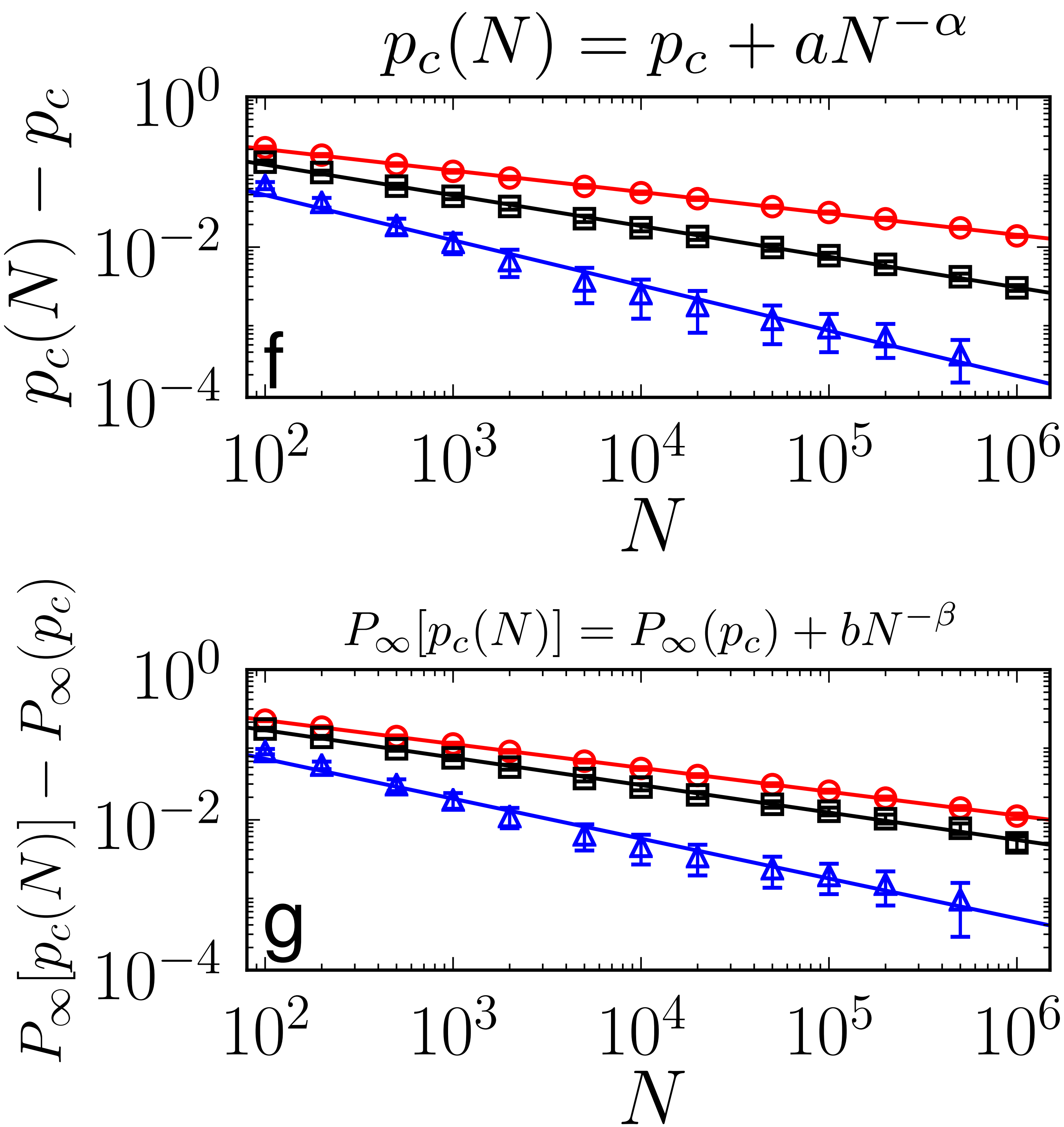
$$P_\infty(p_c)\langle k\rangle = 1.2564$$

F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

Buldyrev, Sergey V., et al. "Catastrophic cascade of failures in interdependent networks." *Nature* 464 (2010): 1025-1028.

Finite size scaling analysis

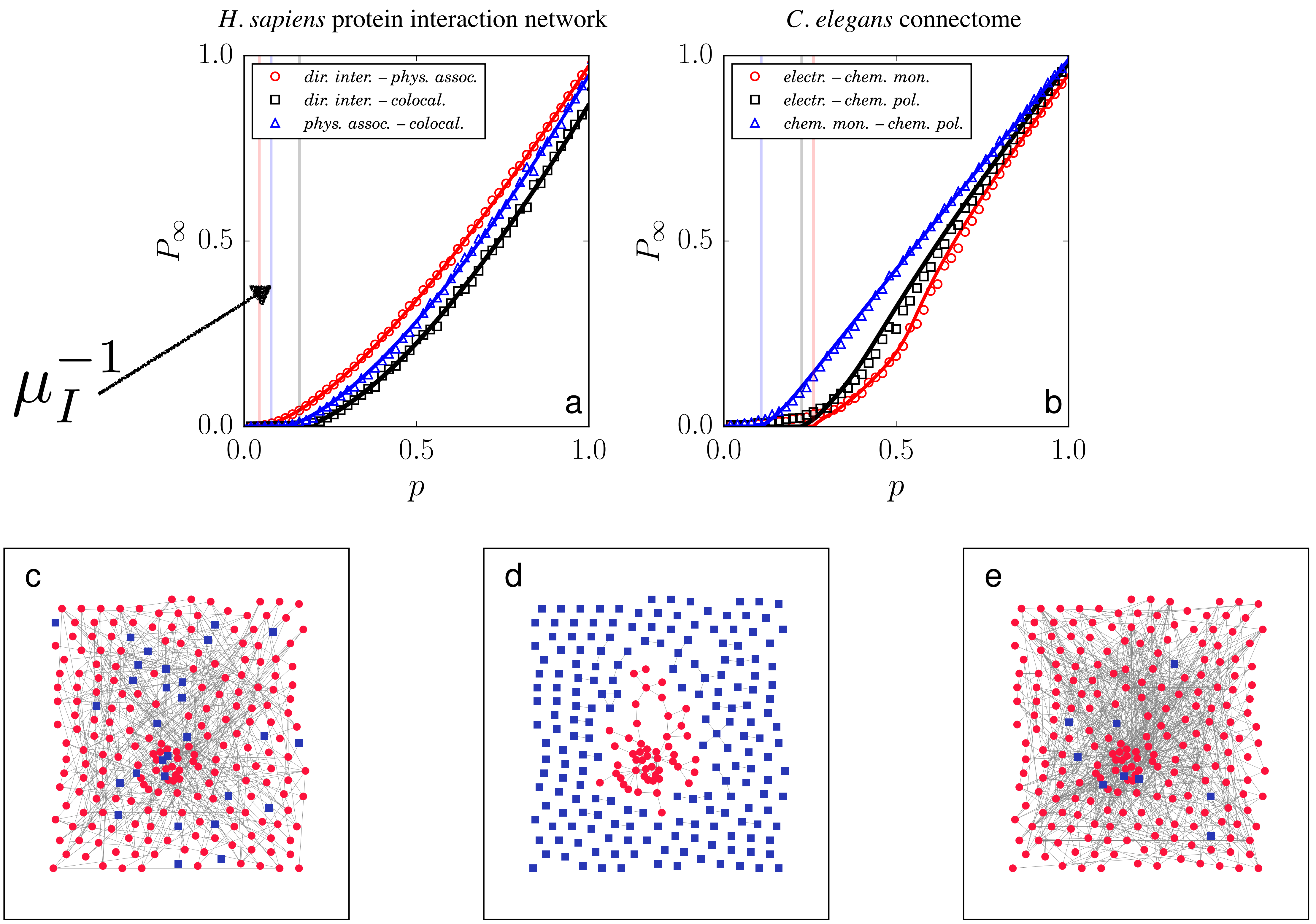
Configuration model, SF graphs



γ	p_c	α	R^2	$P_\infty(p_c)$	β	R^2
2.3	0.17	0.29	1.00	0.01	0.32	1.00
2.7	0.25	0.42	1.00	0.07	0.37	1.00
3.5	0.34	0.68	0.99	0.16	0.61	0.99

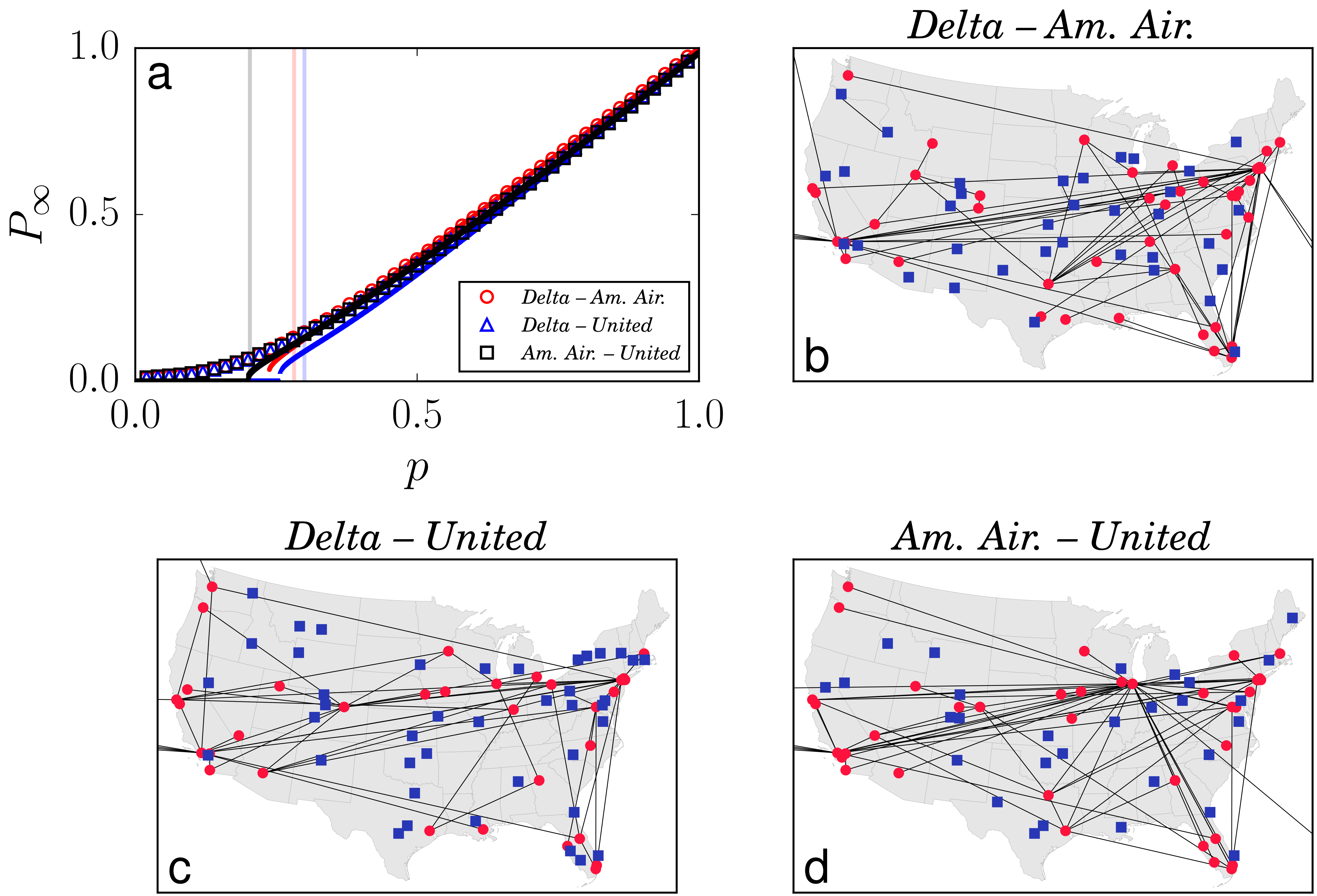
F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

Results on real networks



De Domenico, M., Porter, M. A. & Arenas, A. Muxviz: a tool for multilayer analysis and visualization of networks. *Journal of Complex Networks* cnu038 (2014).

Results on real networks



F. Radicchi, Percolation in real interdependent networks, Nature Physics 11, 597-602 (2015)

Percolation on real multiplex networks

better approximations

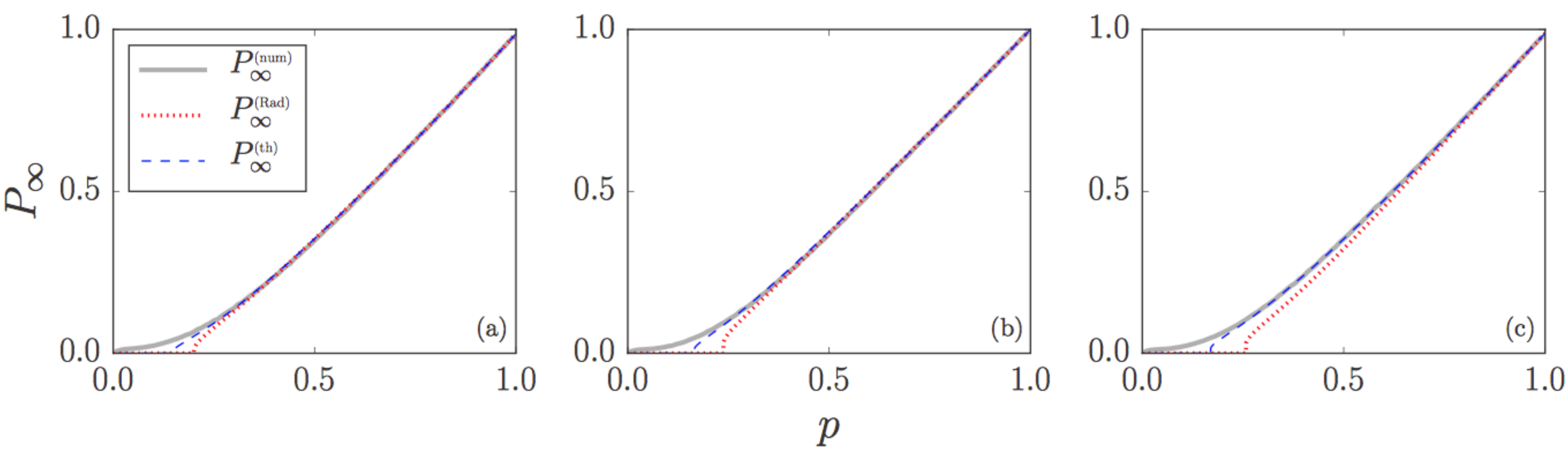
$$s_{i\to j}^{(1,1),(1,1)} = s_{i\to j}^{(1,0),(1,0)} = s_{i\to j}^{(0,1),(0,1)}$$

$$= p\left[1 - \prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[1]}\right) - \prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[2]}\right) + \prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[1,2]}\right)\right], \quad (1)$$

$$s_{i\to j}^{(1,1),(1,0)} = p\left[\prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[2]}\right) - \prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[1,2]}\right)\right], \quad (2)$$

and

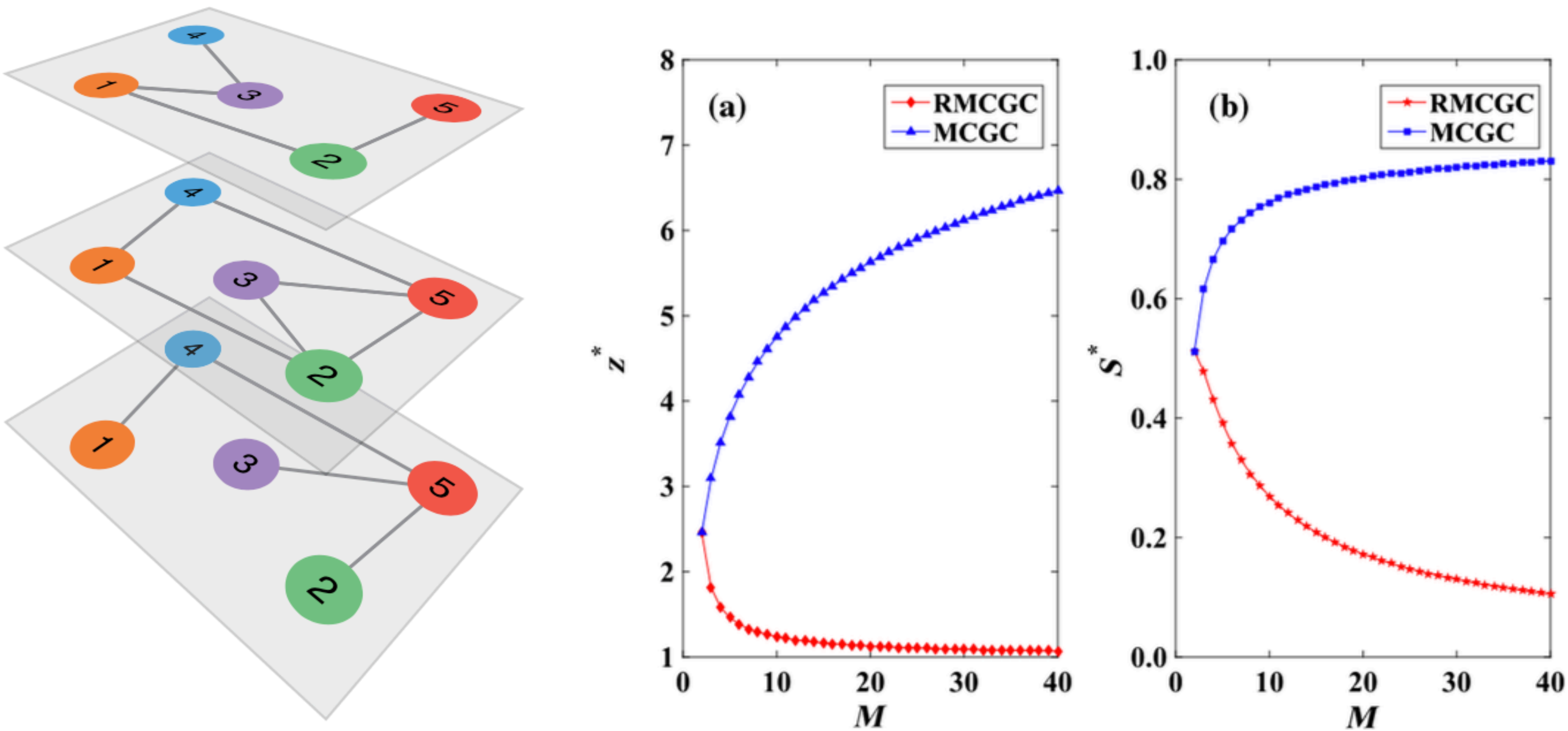
$$s_{i\to j}^{(1,1),(0,1)} = p\left[\prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[1]}\right) - \prod_{\ell\in N(i)\setminus j}\left(1 - z_{\ell\to i}^{[1,2]}\right)\right]. \quad (3)$$



G. Bianconi and F. Radicchi, Phys. Rev. E **94**, 060301(R) (2016)

Redundant percolation on multiplex networks

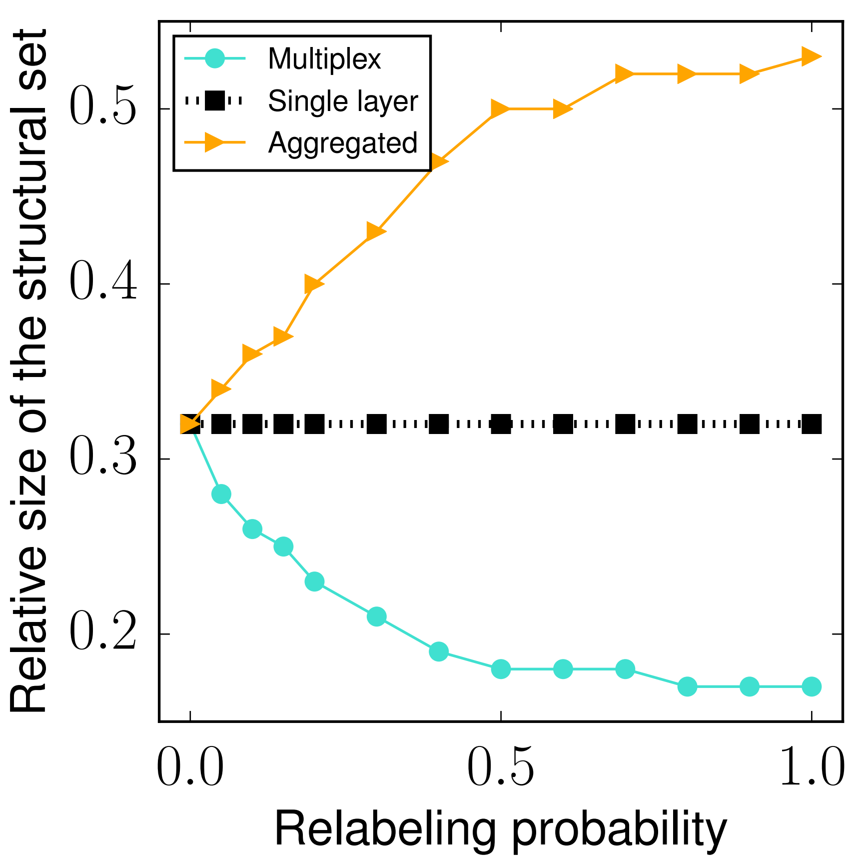
Addition of layers improves robustness

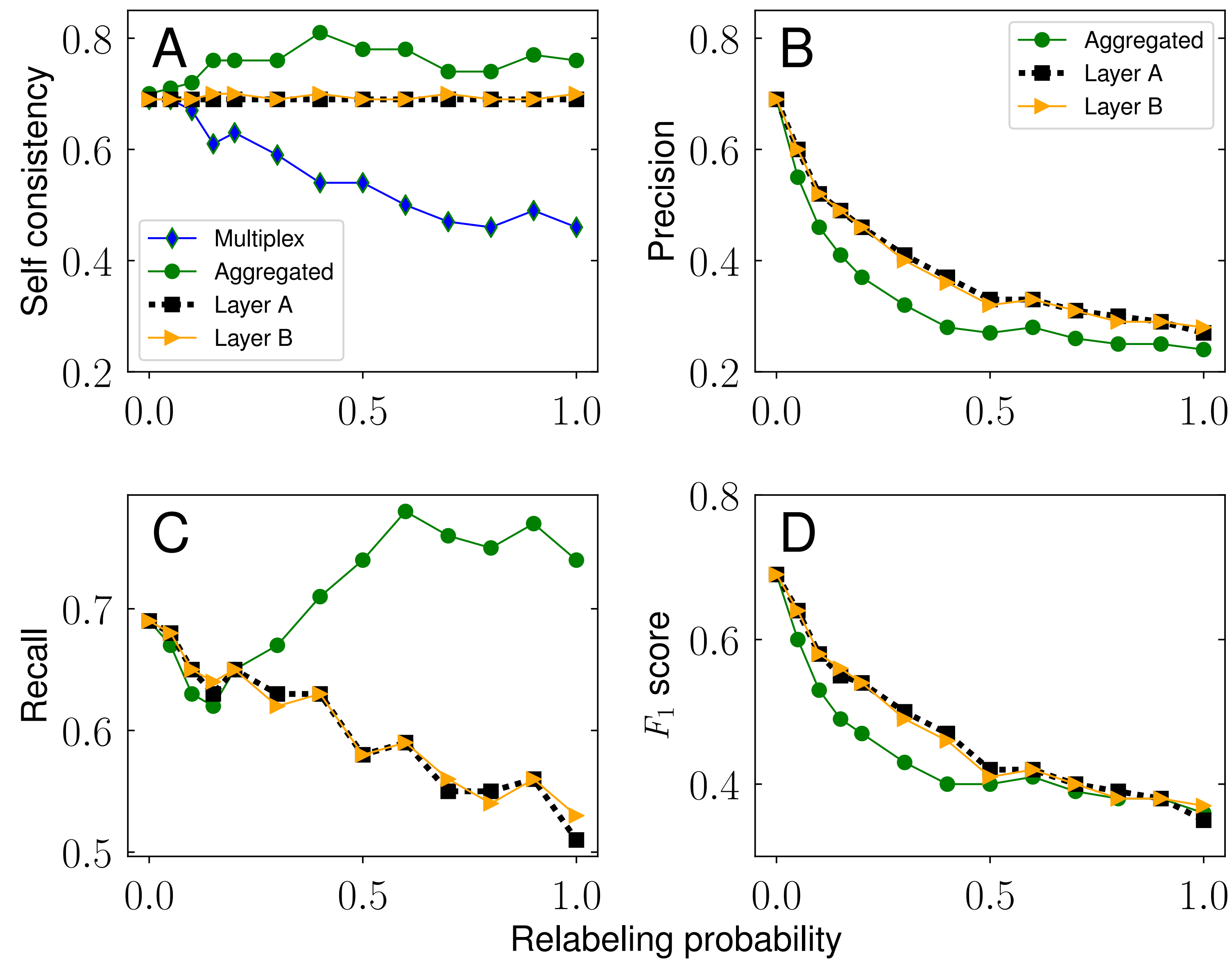


F. Radicchi and G. Bianconi, Phys. Rev. X **7**, 011013 (2017)

Optimal percolation

On multiplex networks

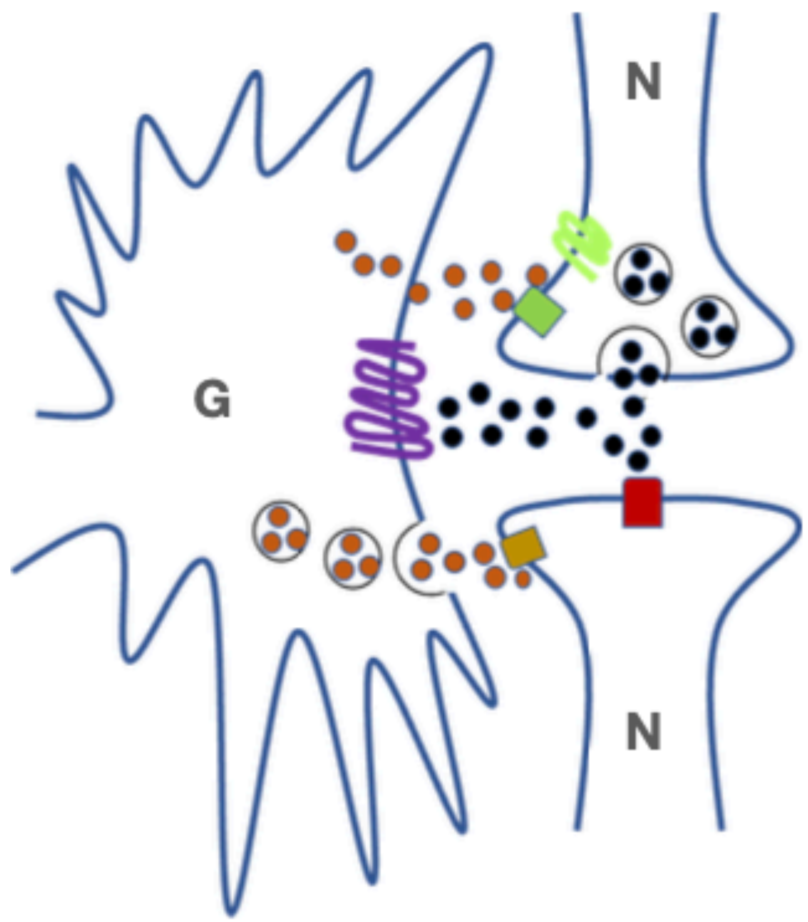


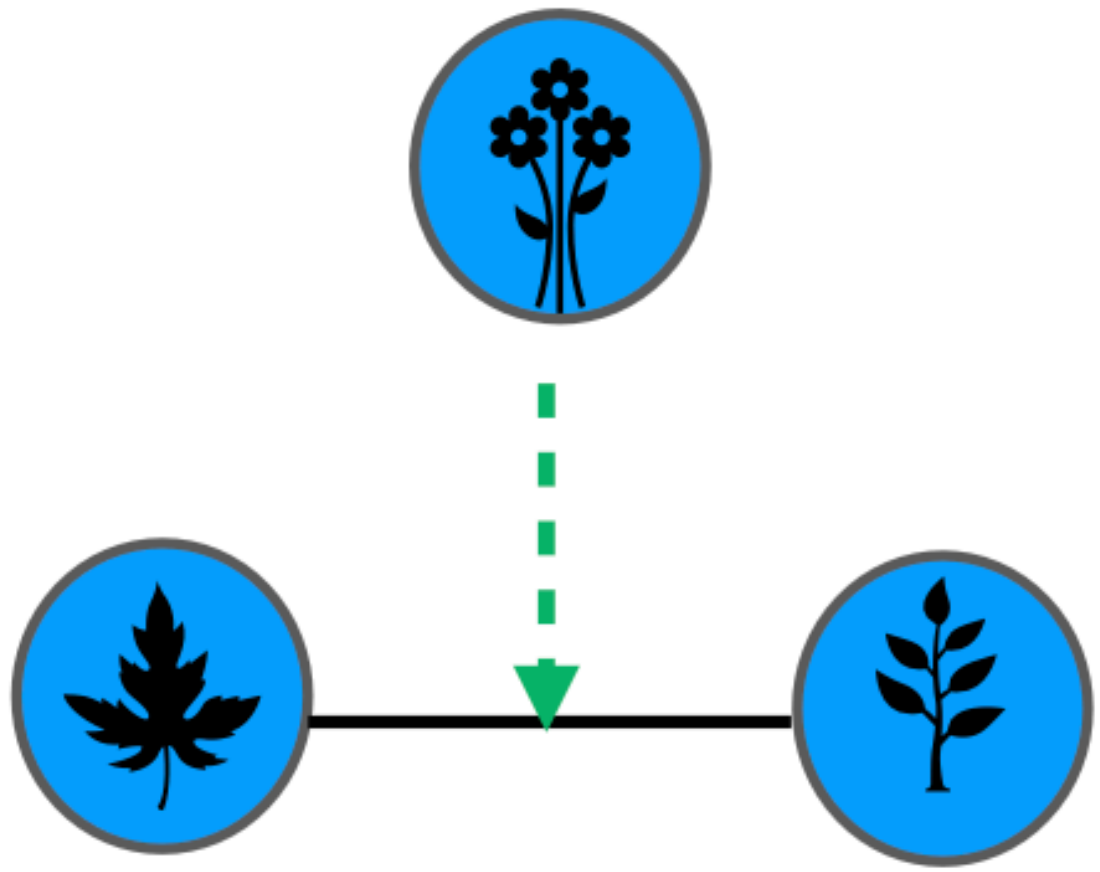


S. Osat, A. Faqeeh and F. Radicchi, Nature Communications **8**, 1540 (2017)

Triadic percolation

a model of percolation in networks with higher-order interactions





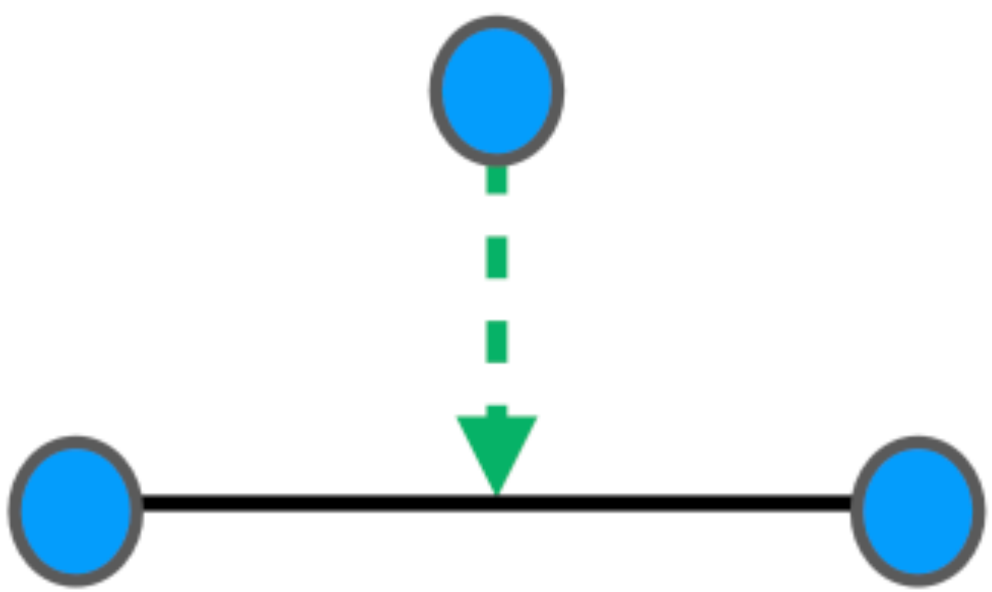
in some real systems, the interaction between two elements is regulated by a third-party element

H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Triadic percolation

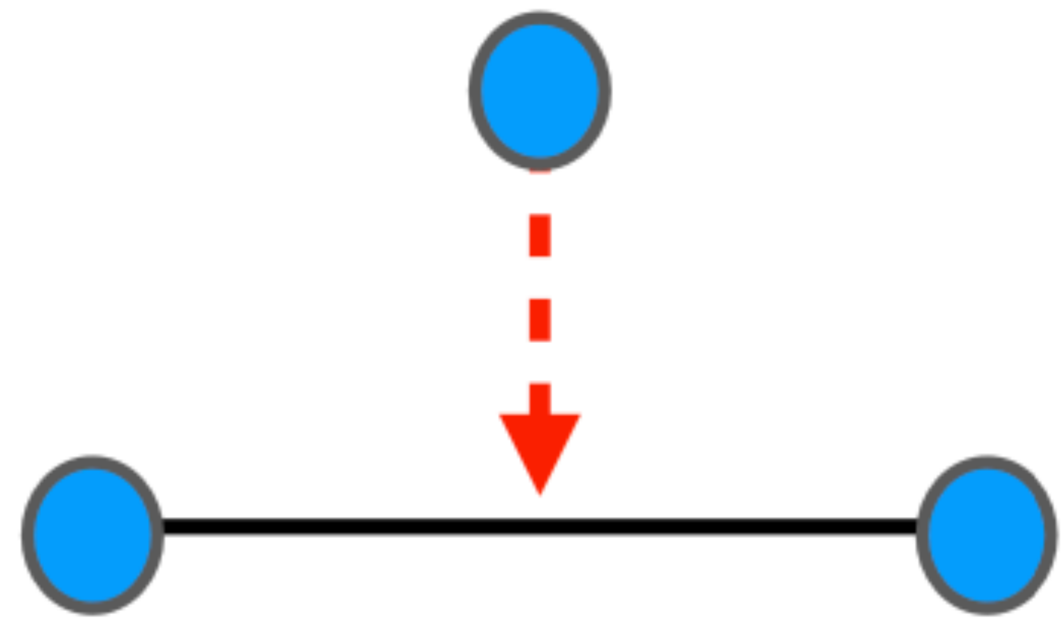
a model of percolation in networks with higher-order interactions

positive regulation



an active node
activates the link that
is regulating

negative regulation



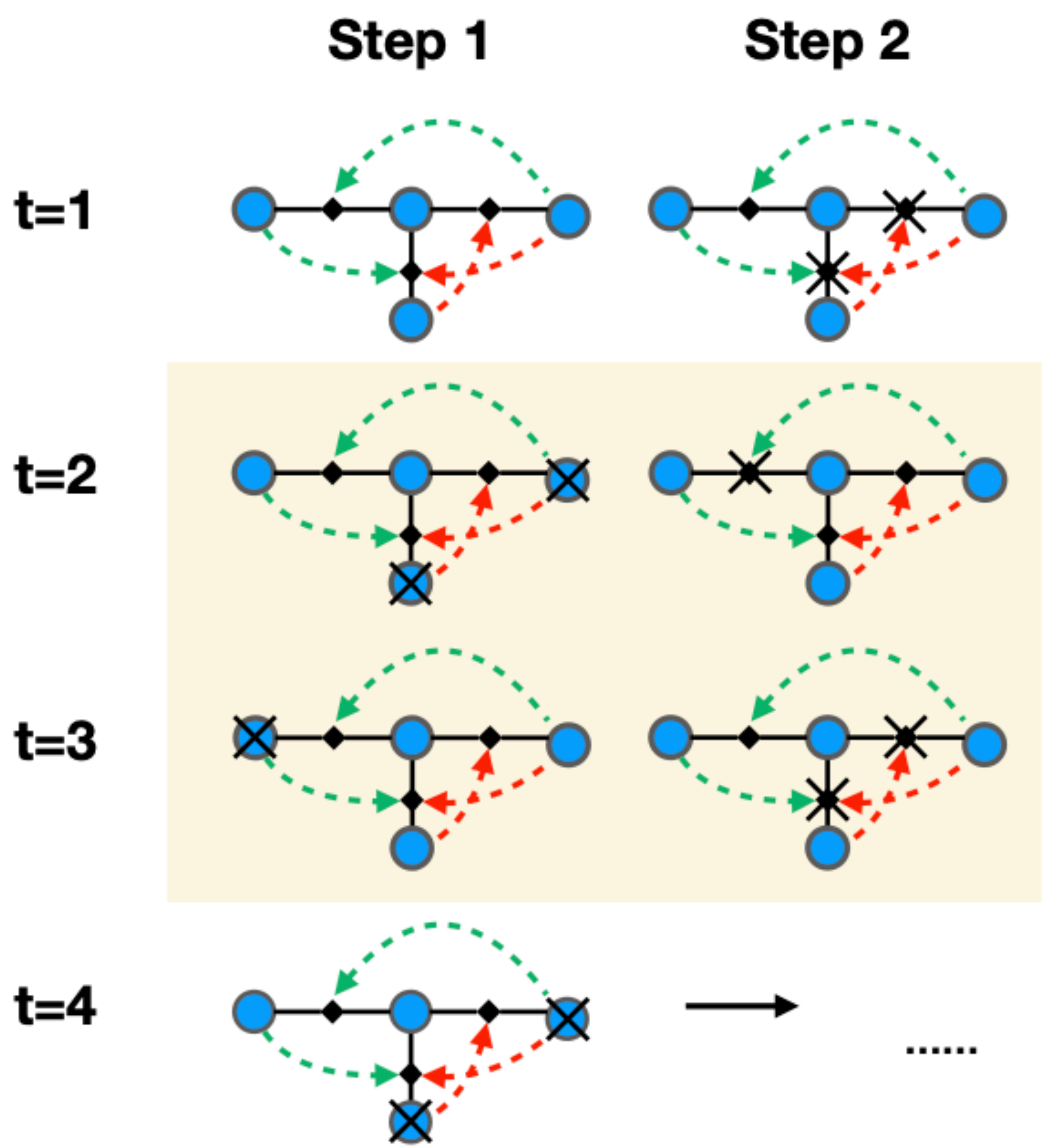
an active node
deactivates the link
that is regulating

a node is active if in the GCC of the network

H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Triadic percolation

a model of percolation in networks with higher-order interactions



H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Triadic percolation

a model of percolation in networks with higher-order interactions

model

1 Given the configuration of activity of the structural links at time t – 1, we define each node active if the node belongs to the GCC of the structural network in which we consider only active links. The node is considered inactive otherwise.

2 Given the set of all active nodes obtained in step 1, we deactivate all the links that are connected at least to one active negative regulator node and/or that are not connected to any active positive regulator node. All the other links are deactivated with probability $q = 1 - p$.

network structure

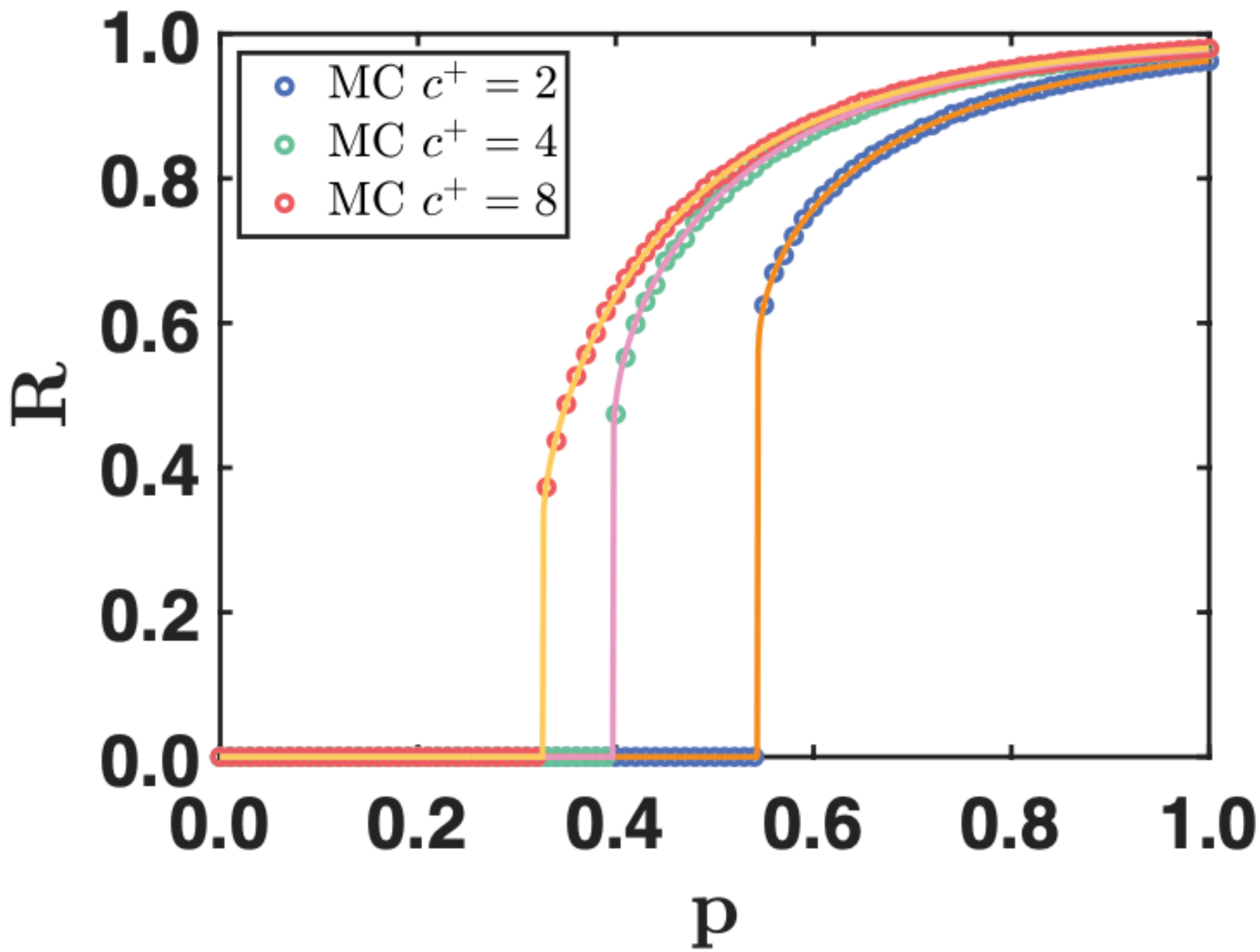
Structural network given by ER model with average degree k

Each node of the structural network acts as positive regular for c+ randomly chosen links and as negative regulator for c- randomly chosen links.

H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Triadic percolation

a model of percolation in networks with higher-order interactions

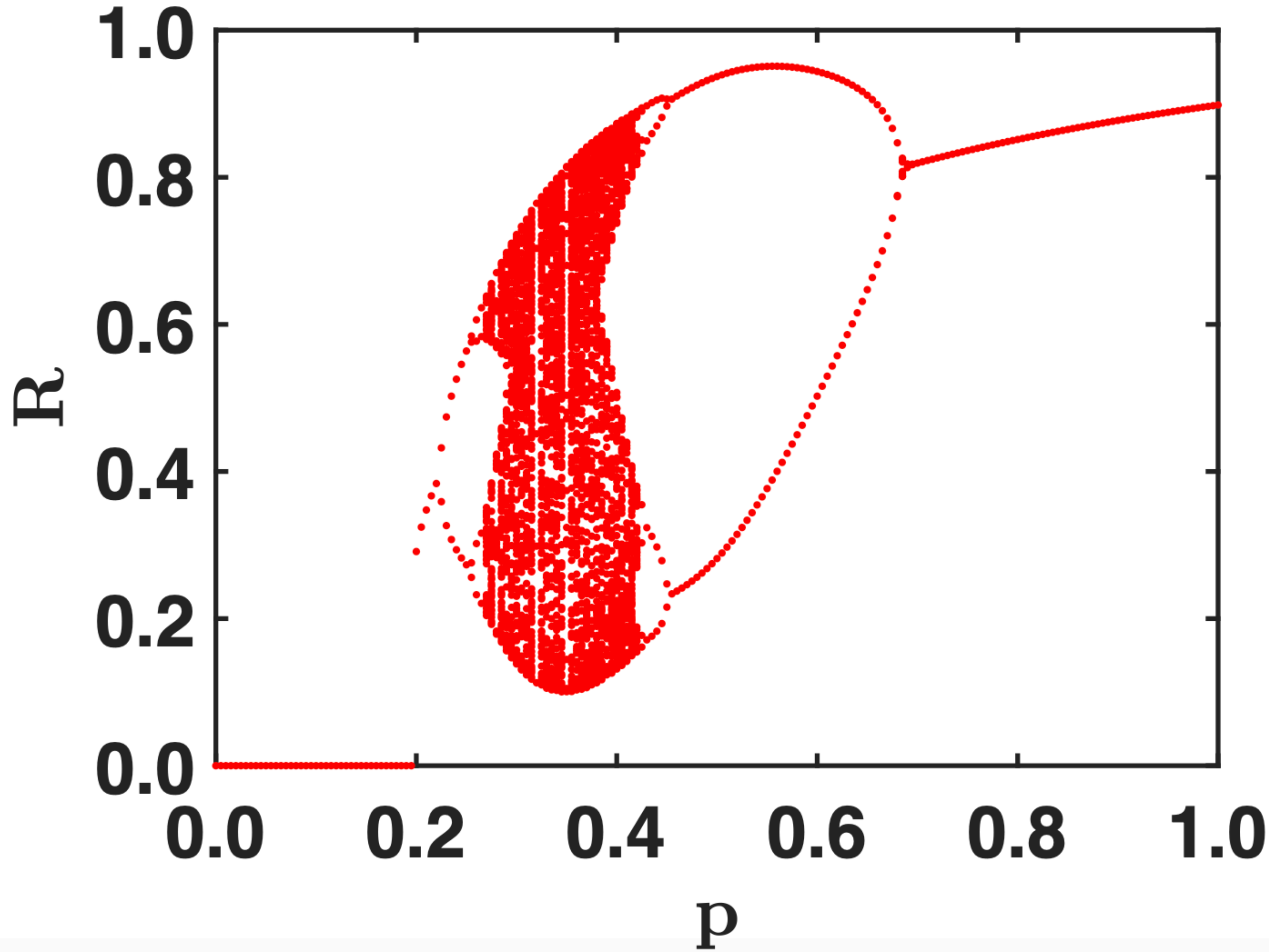


in absence of negative regulator nodes

H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Triadic percolation

a model of percolation in networks with higher-order interactions



in presence of both negative/positive regulator nodes

H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Triadic percolation

a model of percolation in networks with higher-order interactions

The model displays a rich dynamical behavior characterized by period doubling and a route to chaos

For ER structural networks, the model belongs to the same universality class as of the logistic map

H. Sun, F. Radicchi, J. Kurths and G. Bianconi, arXiv:2204.13067 (2022)

Thanks!

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F. Radicchi and A. Arenas
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Breaking of the site-bond percolation universality in networks
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Beyond the locally treelike approximation for percolation on real networks
F. Radicchi and C. Castellano
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Leveraging percolation theory to single out influential spreaders in networks
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Observability transition in real networks
Y. Yang and **F. Radicchi**
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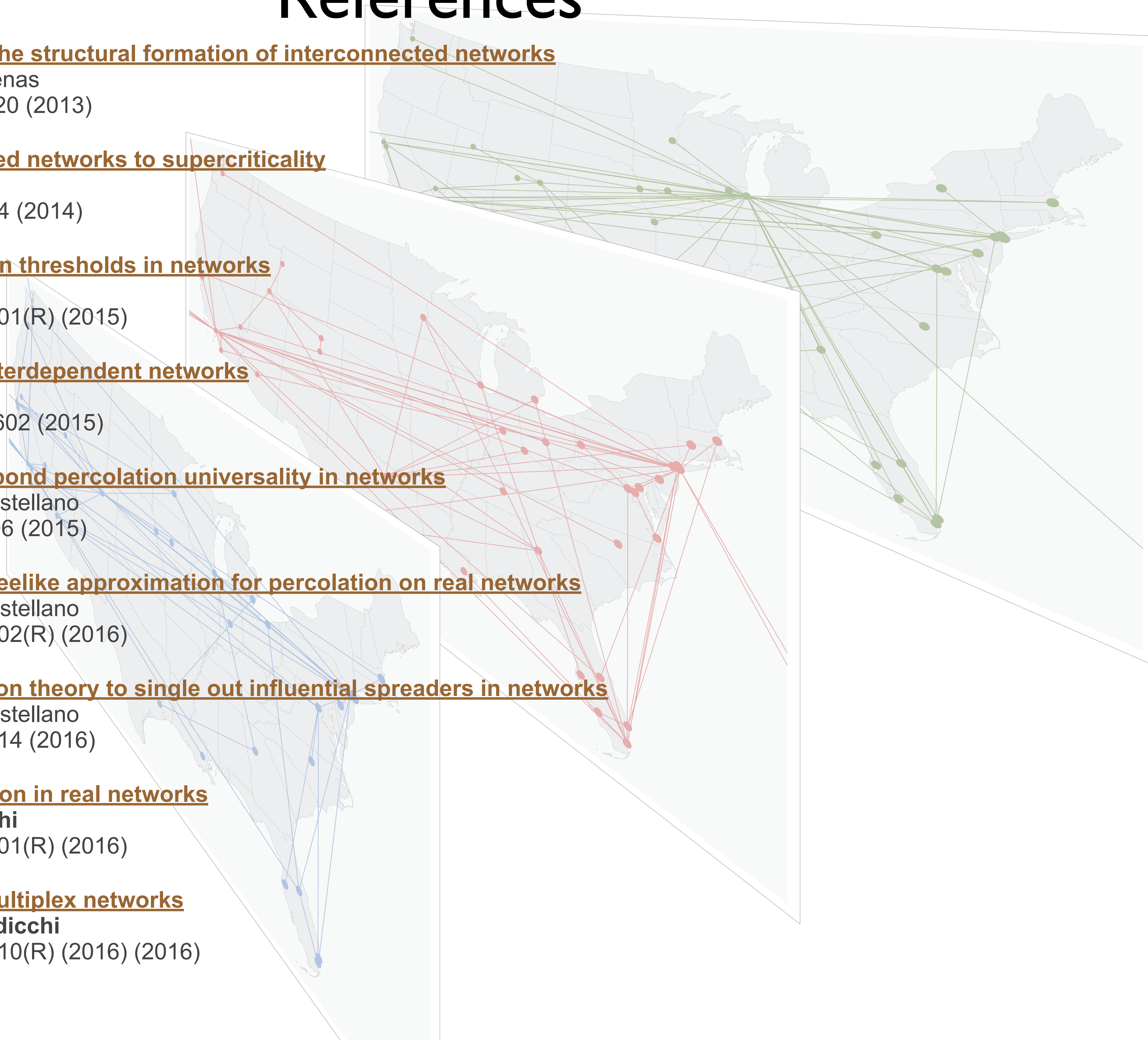
Percolation in real multiplex networks
G. Bianconi and **F. Radicchi**
Phys. Rev. E 94, 060310(R) (2016) (2016)



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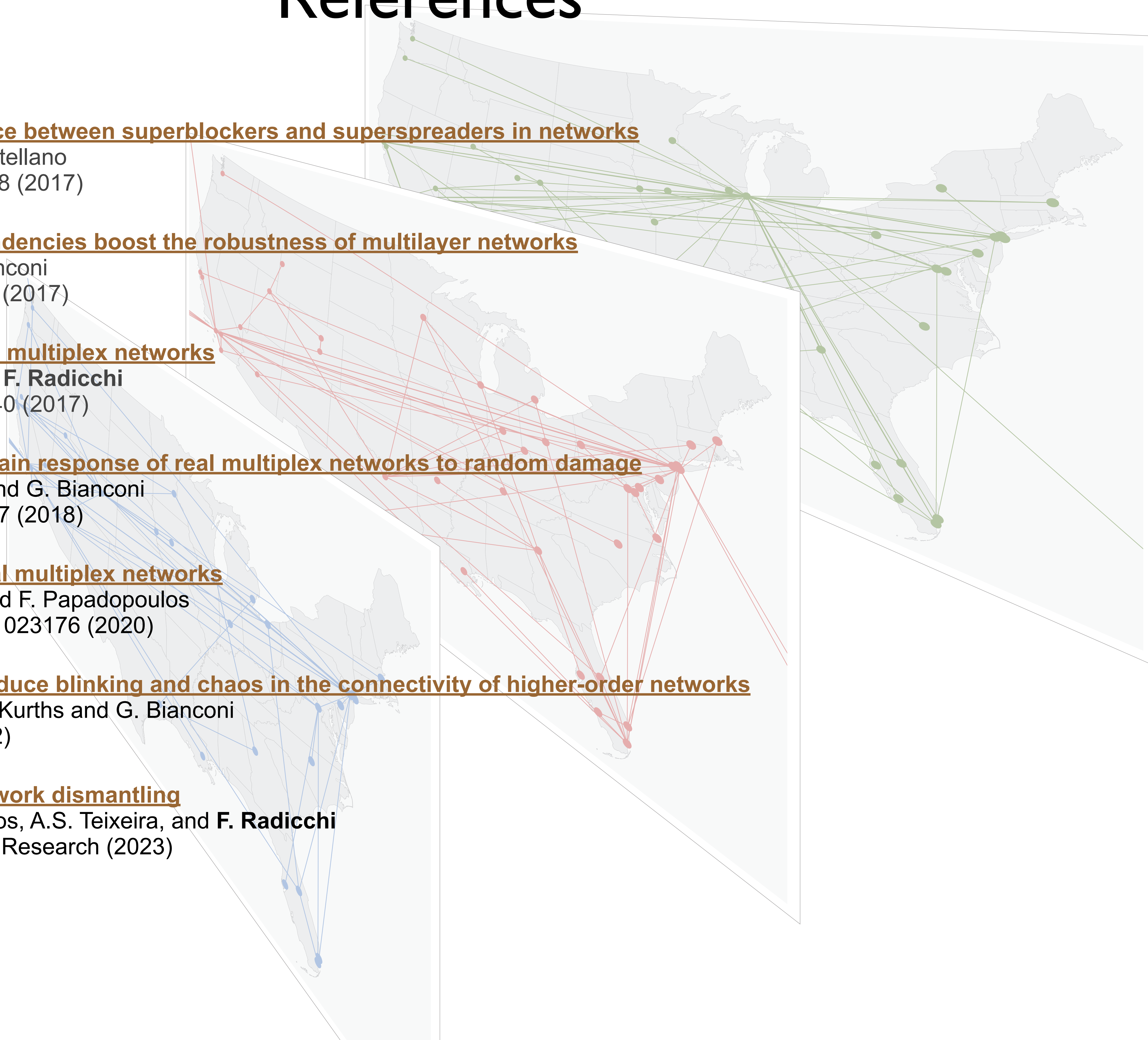

Fundamental difference between superblockers and superspreaders in networks
F. Radicchi and C. Castellano
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Redundant interdependencies boost the robustness of multilayer networks
F. Radicchi and G. Bianconi
Phys. Rev. X **7**, 011013 (2017)

Optimal percolation in multiplex networks
S. Osat, A. Faqeeh and **F. Radicchi**
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Controlling the uncertain response of real multiplex networks to random damage
F. Coghi, **F. Radicchi** and G. Bianconi
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k-core structure of real multiplex networks
S. Osat, **F. Radicchi** and F. Papadopoulos
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Triadic interactions induce blinking and chaos in the connectivity of higher-order networks
H. Sun, **F. Radicchi**, J. Kurths and G. Bianconi
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Embedding-aided network dismantling
S. Osat, F. Papadopoulos, A.S. Teixeira, and **F. Radicchi**
to appear in Phys. Rev. Research (2023)